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Splines with Nonnegative B-Spline Coefficients

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Abstract. We consider the question of the approximation of nonnegative functions by nonnegative splines of order k (degree $< k$) compared with approximation by that subclass of nonnegative splines of order k consisting of all those whose B-spline coefficients are nonnegative; while approximation by the former gives errors of order h^k, the latter may yield only h^2. These results are related to certain facts about quasi-interpolants.

1. Introduction. In approximating or representing certain data by a spline function s subject to continuous inequality constraints such as $s(x) \geqq g(x)$ on $[0, 1]$ for a given function g, the continuous constraints are often difficult to treat numerically. To avoid this problem, one might replace g by an accurate spline approximation s_g and then enforce $s \geqq s_g$ by enforcing the still stronger condition $A_j(s) \geqq A_j(s_g)$, where for any spline s we denote by $A_j(s)$ the jth coefficient in the representation of s as a linear combination of B-splines; since each B-spline is nonnegative, this condition is at least as strong as the condition $s \geqq s_g$. One is immediately forced to ask whether or not the approximation properties of the class of splines satisfying $A_j(s) \geqq A_j(s_g)$ are as good as those for the class satisfying $s \geqq s_g$. In this short note, we take the simple case of $s_g \equiv g \equiv 0$ and demonstrate that much approximation power is lost by this computational simplification. As usual, the results can be extended to higher dimension via tensor products. We have been informed that these results and others, of both a theoretical and a computational nature, have been obtained independently by John Lewis and will eventually appear in his Ph.D. dissertation in computer science at Yale University.

2. A Constrained Approximation Problem. We shall consider the problem of approximating a given *nonnegative* function f on a closed and bounded interval $[a, b]$ by splines of positive integer order k (degree $< k$); unbounded and open intervals can be treated similarly. Let $\pi := (x_i)_0^N$ be a k-extended partition of $[a, b]$, so that $a = x_0 < x_1 \leqq x_2 \leqq \cdots \leqq x_{N-1} < x_N = b$ with no more than k consecutive x_j's coinciding, i.e., $x_j < x_{j+k}$. Let $S_\pi{}^k$ denote the linear space of polynomial splines of order k on π, so that each s in $S_\pi{}^k$ is a polynomial of degree less than k in each interval (x_i, x_{i+1}) and $s^{(r)}$ is continuous at x_i for all $r < k - d_i$, where d_i is the frequency with which the number $x = x_i$ appears among the x_j's. Then the sequence

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of normalized B-splines $\{N_{j,k} \mid j = -k+1, -k+2, \cdots, N-1\}$ is a basis for $S_\pi{}^k$, where we have augmented π by additional points $x_{-k+1} = \cdots = x_{-1} = a$ and $b = x_{N+1} = \cdots = x_{N+k-1}$. As usual (Curry-Schoenberg [4], de Boor [1]), $N_{i,k}$ denotes the normalized B-spline defined by divided differences: $N_{i,k}(x) := (x_{i+k} - x_i)g_k(x_i, \cdots, x_{i+k}; x)$ with $g_k(t; x) = (t-x)_+^{k-1}$. These B-splines are so normalized that $\sum_i N_{i,k}(x) \equiv 1$. Since these B-splines are also nonnegative, it follows tha.

$$\left|\sum_i A_i N_{i,k}(x)\right| \leqq \max_i |A_i|. \tag{2.1}$$

We wish to approximate the nonnegative function f on $[a, b]$ by an element s in $S_\pi{}^k$ with the constraint that $A_j(s) \geqq 0$ for all j, where $s = \sum_j A_j(s)N_{j,k}$. This, of course, implies that $s(x) \geqq 0$ on $[a, b]$ since every $N_{i,k}$ is nonnegative; in fact, since for $k > 2$ the B-splines do not give all the extreme points in the cone of nonnegative splines (Burchard [3]), the condition $A_j(s) \geqq 0$ for all j is appreciably stronger than $s \geqq 0$. We shall see just how much stronger it is.

For any f in the Sobolev space $W_\infty{}^q[a, b]$ with $1 \leqq q \leqq k$, we recall the definition (de Boor-Fix [2]) of the quasi-interpolant $F_{\pi,q}f \in S_\pi{}^k$ of f,

$$F_{\pi,q}f = \sum_i \lambda_{i,q}(f)N_{i,k} \tag{2.2}$$

where

$$\lambda_{j,q}(f) = \sum_{r<q} (-1)^{k-1-r}\psi_j{}^{(k-1-r)}(\tau_j)f^{(r)}(\tau_j) \tag{2.3}$$

and

$$\psi_j(x) = (x_{j+1} - x) \cdots (x_{j+k-1} - x)/(k-1)!. \tag{2.4}$$

We choose τ_j to satisfy $x_j{}^+ \leqq \tau_j \leqq x_{j+k}{}^-$ as well as $\tau_j \in [a, b]$. Then one has (de Boor-Fix [2])

$$||f - F_{\pi,q}f||_\infty \leqq K_0\, ||f^{(q)}||_\infty\, |\pi|^q, \qquad |\pi| = \max_i |x_{i+1} - x_i| \tag{2.5}$$

for some constant K_0 independent of f and π. Also, for $q = k$ and $s \in S_\pi{}^k$, one has

$$\lambda_{j,k}(s) = A_j(s) \quad \text{and, therefore,} \quad F_{\pi,k}s = s. \tag{2.6}$$

From Eq. (2.3), we have

$$\begin{aligned}\lambda_{j,q}(f) &= f(\tau_j) + (-1)^{k-2}\psi_j{}^{(k-2)}(\tau_j)f^{(1)}(\tau_j) + \sum_{r=2}^{q-1} (-1)^{k-1-r}\psi^{(k-1-r)}(\tau_j)f^{(r)}(\tau_j) \\ &= f(\tau_j) + (-1)^{k-2}\psi_j{}^{(k-2)}(\tau_j)f^{(1)}(\tau_j) + O(|\pi|^2).\end{aligned} \tag{2.7}$$

Since ψ is of degree less than k and vanishes at $x_{j+1}, \cdots, x_{j+k-1}$, we know that $\psi^{(k-2)}$ vanishes at the point $\xi_j = (x_{j+1} + \cdots + x_{j+k-1})/(k-1)$ in (x_{j+1}, x_{j+k-1}); choose $\tau_j = \xi_j$. Then, from Eq. (2.7), we have $\lambda_{j,q}(f) = f(\tau_j) + O(|\pi|^2)$ if $f^{(2)}, \cdots, f^{(q-1)}$ are all bounded. Since f is nonnegative, a shift in the B-spline coefficients by $O(|\pi|^2)$ produces a new approximation to f with nonnegative B-spline coefficients and, by Eq. (2.1), only $O(|\pi|^2)$ away from the original high-order approximation. This proves the following:

(2.8) PROPOSITION. *Let $f \in W_\infty^2[a, b]$ be nonnegative, and let $k \geqq 2$. Then there exists a spline s_p in S_π^k with nonnegative B-spline coefficients and such that $\|s_p - f\|_\infty = O(|\pi|^2)$. Equivalently, there exists a quasi-interpolant s whose B-spline coefficients A_i converge to values of f in $[x_i, x_{i+k}]$ at least as fast as $O(|\pi|^2)$.*

It should be noted that $F_{\pi,2}$, with the τ_i's as chosen above, is just Schoenberg's variation diminishing approximation scheme (Schoenberg [6]) studied in (Marsden [5]) from which Proposition 2.8 could also be derived.

We now proceed to show that the result in Proposition 2.8 is essentially best possible; since this is clear for $k = 2$, we consider $k > 2$. Consider the function $f(x) = x^2$ on the interval $[-1, 1]$, and consider *uniform* partitions π_n of width $|\pi_n| = h = 2/n$ for integers n. If k is even, we consider partitions π_n for which n is also even; for k odd, we consider only n odd.

We now provide details for the case in which $k = 2l + 2$ is even and $n = 2m$; the argument in the remaining case is similar. Let $j = m - l - 1$ and consider $\lambda_{j,k}(f)$ with $\tau_j \equiv 0$. Since $f(\tau_j) = f^{(1)}(\tau_j) = f^{(r)}(\tau_j) = 0$ for $r > 2$ and $f^{(2)}(\tau_j) = 2$, we conclude from Eq. (2.3) that $\lambda_{j,k}(f) = (-1)^{k-3}\cdot 2\cdot \psi_j^{(k-3)}(0)$. By simple induction, one finds that

$$\psi_j^{(k-3)}(0) = \frac{h^2}{2l(2l-1)} \sum_{i=1}^{l} i^2 = \frac{l+1}{12} h^2,$$

so that

$$\lambda_{m-l-1,k}(f) = -\frac{l+1}{6} h^2. \tag{2.9}$$

It is known (de Boor [1]) that there exists a constant $D_k < \infty$ and independent of π such that $|\lambda_{i,k}(s)| \leqq D_k\|s\|_\infty$ for all i and for all s in S_π^k. Note that $f(x) = x^2$ is in S_π^k for $k > 2$. Let s_p be any spline in S_π^k with nonnegative B-spline coefficients, i.e., satisfying $\lambda_{i,k}(s_p) \geqq 0$ for all i. Then

$$\begin{aligned} 0 < -\lambda_{m-l-1,k}(f) &= \lambda_{m-l-1,k}(-f) \leqq \lambda_{m-l-1,k}(-f) + \lambda_{m-l-1,k}(s_p) \\ &= \lambda_{m-l-1,k}(s_p - f) \leqq D_k \|s_p - f\|_\infty. \end{aligned}$$

Therefore, $\|s_p - f\|_\infty \geqq -D_k^{-1}\lambda_{m-l-1,k}(f)$; using Eq. (2.9), we now see that Proposition (2.8) is essentially best possible.

(2.10) PROPOSITION. *For $f(x) = x^2$ on $[-1, 1]$, π the uniform partition of width $1/m$, and $k = 2l + 2$, every spline s_p in S_π^k with nonnegative B-spline coefficients satisfies $\|s_p - f\|_\infty \geqq D_k^{-1}|\pi|^2((l + 1)/6)$. Equivalently, the B-spline coefficients converge to a value of f no faster than $O(|\pi|^2)$.*

Since it is easy to approximate a nonnegative function $f \in W_\infty^q$ by nonnegative splines to within $O(|\pi|^q)$ simply by translating $F_{\pi,q}f$ by $O(|\pi|^q)$, we see that approximation by splines with nonnegative B-spline coefficients may lose much of the approximation power of splines. By arguing essentially as we did for $f(x) = x^2$, it is easy to show that for general nonnegative $f \in W_\infty^k$ the precise order of best approximation by elements of S_π^k with nonnegative B-spline coefficients is $O(|\pi|^k)$ plus the order of the most negative B-spline coefficient in $F_{\pi,k}f$; as we saw for $f(x) = x^2$, this total error can be as large as $O(|\pi|^2)$, but it is no larger.

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