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L^p Approximation of Fourier Transforms and Certain Interpolating Splines

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Abstract. We extend to L^p, $1 \leqq p \leqq \infty$, the L^2 results of Bramble and Hilbert on convergence of discrete Fourier transforms and on approximation using smooth splines. The main tools are the estimates of [1] for linear functionals on Sobolev spaces and elementary results on Fourier multipliers.

1. Introduction. The purpose of this paper is to extend to L^p, $1 \leqq p \leqq \infty$, the L^2 results of [1] on convergence of discrete Fourier transforms and on spline interpolation. An estimate for a linear functional on the Sobolev space $H_m^p(R^N)$ is fundamental to the work. Multipliers for Fourier transforms provide the general setting for the L^p estimates.

In Section 2, we give definitions and notation and prove the estimate for the linear functional. We study, in Section 3, the difference between discrete and continuous Fourier transforms. In Section 4, we consider certain spline interpolating functions on uniform meshes in R^N. In Section 5, we mention improvements in the L^p estimates for $1 < p < \infty$.

Silliman [5] has obtained the Corollary to Theorem 11 in Section 4 for $p = 1$, $N = 1$, $\beta = 0$, and the spline interpolant Su of order $2m$. We wish to thank Professor M. A. Jodeit for a useful suggestion, and the referee for an interesting comment.

2. Preliminaries. Let Ω be a cube in R^N. For $1 \leqq p < \infty$, $L^p(\Omega)$ denotes the space of functions u defined on Ω for which

$$\|u\|_{p,\Omega} = \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}$$

is finite. L^p is the usual space of functions on all of R^N. $C(\Omega)$ denotes the space of continuous functions u on Ω, and

$$\|u\|_{\infty,\Omega} = \sup_{x \in \Omega} |u(x)|.$$

We define L_0^∞ to be the space of continuous functions u on R^N such that $u(x) \to 0$ as $|x| \to \infty$, and define

$$\|u\|_\infty = \sup_{x \in R^N} |u(x)|.$$

Obviously, L_0^∞ is a Banach space. For $j = 1, \ldots, N$, we write $D_j = -i\partial/\partial x_j$, and

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for $\alpha = (\alpha_1, \ldots, \alpha_N)$ we write $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. For $p < \infty$, H_m^p denotes the space of functions u such that $D^\alpha u \in L^p$ for all α with $|\alpha| \leqq m$. We define

$$\|u\|_{p,m} = \sum_{|\alpha| \leqq m} \|D^\alpha u\|_p.$$

We define H_m^∞ to be the space of functions u such that $D^\alpha u \in L_0^\infty$ for all α with $|\alpha| \leqq m$. We define

$$\|u\|_{\infty,m} = \sum_{|\alpha| \leqq m} \|D^\alpha u\|_\infty.$$

P_k denotes the space of polynomials of total degree k.

We define the Fourier transform of a function u on R^N as

$$\hat{u}(\xi) = (2\pi)^{-N/2} \int u(x) e^{-i\langle x,\xi\rangle}\, dx$$

where $\langle x, \xi\rangle = \sum_{j=1}^N x_j \xi_j$. $\check{u}$ denotes the inverse Fourier transform of u. For $h > 0$, Z_h^N denotes the set $\{\mu h\colon \mu = (\mu_1, \ldots, \mu_N),\ \mu_j \text{ an integer}\}$. Define $Q_h = \{\xi \in R^N\colon -\pi < h\xi_j \leqq \pi, j = 1, \ldots, N\}$. χ_h denotes the characteristic function of Q_h. It is well known that, if μ is a measure of finite total variation on R^N, then, for $1 \leqq p \leqq \infty$ and for all $u \in L^p$,

$$\|\mu * u\|_p \leqq C\|u\|_p,$$

where $(\mu * u)(x) = \mu(u(x - \cdot))$. The following result is easily established using Parseval's identity and the Cauchy-Schwarz inequality.

THEOREM 1. *Suppose $s > N/2$ and $f \in H_m^2$. Then $\hat{f} \in L^1$.*

Let $\psi = \chi_{2\pi}$ and $\psi^{*k} = \psi * \cdots * \psi$ where there are k ψ's. Define the operator E by

$$Eu(x) = \sum_{\mu \in Z^N} \psi(x - \mu) u(\mu).$$

For $\xi \in R^N$, define $\sin \xi = (\sin \xi_1, \ldots, \sin \xi_N)$ and $\Phi(\xi) = \prod_{j=1}^N \xi_j$. Then

$$\hat{\psi}(\xi) = (2\pi)^{-N/2} \Phi(2 \sin \xi/2) \Phi^{-1}(\xi). \tag{2.0}$$

For a fixed $x \in R^N$, define the linear functional $F_m(x, \cdot)$ on C_0^∞ by

$$F_m(x, u) = (\psi^{*m-1} * Eu)(x) - (\psi^{*m} * u)(x).$$

For r a polynomial, we consider $F_m(x, r)$ to be a tempered distribution.

LEMMA 2. *For $r \in P_{m-1}$, $F_m(x, r) = 0$.*

Proof. Write $r(x) = \sum_{|\alpha|<m} a_\alpha x^\alpha$. Let u be a test function. Then a straightforward calculation shows that

$$F_m(\cdot, r)(u) = (2\pi)^{N/2} \sum_\alpha a_\alpha \left\{ \sum_{\mu \in Z^N} [D^\alpha (\psi^{*m})\hat{}\, \check{u}]\hat{}\,(\mu) - \int [D^\alpha (\psi^{*m})\hat{}\, \check{u}]\hat{}\,(y)\, dy \right\}$$

and thus it follows from the Poisson summation formula that

$$F_m(\cdot, r)(u) = (2\pi)^N \sum_{|\alpha|<m} a_\alpha \sum_{\mu \neq 0} [D^\alpha (\psi^{*m})\hat{}\, \check{u}](2\pi\mu). \tag{2.1}$$

Since $(\psi^{*m})^{\wedge}(\xi) = (2\pi)^{-N/2}\Phi^m(2\sin\xi/2)\Phi^{-m}(\xi)$, each term with $\mu \neq 0$ in the right side of (2.1) is 0.

LEMMA 3. *Let* $1 \leqq p \leqq \infty$ *and* $m > N/p$. *There is a constant* C *such that for all* u *in* C_0^∞, *we have*

$$\|F_m(\cdot, u)\|_p \leqq C \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{2.2}$$

Proof. Let Q denote $Q_{2\pi}$ and let $x \in \mu + Q$ with $\mu \in Z^N$. Using the Sobolev imbedding theorem, we have

$$|F_m(x, u)| \leqq C \sum_{|\alpha| \leqq m} \|D^\alpha u\|_{p,\mu+(m+1)Q}$$

where $rQ = \{rx : x \in Q\}$. Applying the Bramble-Hilbert lemma [1, Theorem 2], we obtain

$$|F_m(x, u)| \leqq C \sum_{|\alpha|=m} \|D^\alpha u\|_{p,\mu+(m+1)Q} \tag{2.3}$$

in view of Lemma 2. Taking pth powers and integrating, we have

$$\|F_m(\cdot, u)\|^p_{p,\mu+Q} \leqq C \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\mu+(m+1)Q}. \tag{2.4}$$

Summing on μ, we obtain (2.2).

Throughout this paper, let

$$\eta(\xi) = \prod_{j=1}^{N} a(\xi_j),$$

where a is a C_0^∞ function which vanishes outside $(-\pi, \pi)$ and is 1 on $[-\pi/2, \pi/2]$. Write $\eta_h(\xi) = \eta(h\xi)$.

3. Discrete and Continuous Fourier Transforms. For u a function on Z_h^N with bounded support, we define its discrete Fourier transform $\tilde{u}$ by

$$\tilde{u}(\xi) = (2\pi)^{-N/2} h^N \sum_{\mu \in Z^N} u(h\mu) e^{-ih\langle\mu,\xi\rangle}.$$

We shall compare the discrete and continuous Fourier transforms as $h \to 0$. We first consider only C_0^∞ functions.

LEMMA 4. *Let* $1 \leqq p \leqq \infty$ *and* $m > N/p$. *There is a constant* C *such that for* u *in* C_0^∞, *for* $0 \leqq k \leqq m$, *and for* $0 < h \leqq 1$,

$$\|(\eta_h \tilde{u})^{\vee} - u\|_{p,k} \leqq Ch^{m-k} \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{3.1}$$

Proof. It is sufficient to prove (3.1) with $h = 1$, since a change of variables will give the general result. Let $u \in C_0^\infty$. Then

$$(Eu)^{\wedge}(\xi) = (2\pi)^{N/2}\hat{\psi}(\xi)\tilde{u}(\xi) = \Phi(2\sin\xi/2)\Phi^{-1}(\xi)\tilde{u}(\xi). \tag{3.2}$$

We shall prove (3.1) first for $k = 0$. Obviously,

$$\eta\tilde{u} - \hat{u} = (\tilde{u} - \hat{u})\eta - (1 - \eta)\hat{u}. \tag{3.3}$$

In order to handle the second term on the right in (3.3), we shall use the following identity, which is easily established:

$$\prod_{j=1}^{N} b_j - \prod_{j=1}^{N} d_j = \sum \left(\prod_{j \in J} d_j \right) \prod_{k \in K} (b_k - d_k), \tag{3.4}$$

where the sum is over all choices of disjoint sets J and K such that $J \cup K = \{1, \ldots, N\}$ and $K \neq \varnothing$. Using the definition of η and (3.4), we have

$$(1 - \eta)\hat{u} = \sum \left[\prod_{j \in J} a(\xi_j) \right] \left\{ \prod_{k \in K} [1 - a(\xi_k)] \right\} \hat{u}. \tag{3.5}$$

Let $K = L \cup \{l\}$ in the sum. Then

$$(1 - \eta)\hat{u} = \sum \left[\prod_{j \in J} a(\xi_j) \right] \left\{ \prod_{k \in L} [1 - a(\xi_k)] \right\} \{[1 - a(\xi_l)]/\xi_l^m\} (D_l^m u)^{\wedge}.$$

Since $1 - a$ is a uniformly bounded continuous function, $1 - a$ is the Fourier transform of a bounded measure. It also follows from Theorem 1 that $[1 - a(\xi_l)]/\xi_l^m$ is the Fourier transform of an L^1 function. Thus, for $1 \leqq p \leqq \infty$,

$$\|[(1 - \eta)\hat{u}]^{\vee}\|_p \leqq C \sum_{k=1}^{N} \|D_k^m u\|_p. \tag{3.6}$$

Now, consider the first term on the right in (3.3). Using (3.2), we have

$$\begin{aligned}(\tilde{u} - \hat{u})\eta &= [(2\pi)^{-N/2}(Eu)^{\wedge}\hat{\psi}^{-1} - \hat{u}]\eta \\ &= [(2\pi)^{-N/2}(Eu)^{\wedge}\hat{\psi}^{m-1} - \hat{\psi}^m\hat{u}]\hat{\psi}^{-m}\eta \\ &= [F_m(\cdot, u)]^{\wedge}\Phi^m(\xi)\Phi^{-m}(2 \sin \xi/2)\eta.\end{aligned}$$

Obviously, $\Phi^m(\xi)\Phi^{-m}(2 \sin \xi/2)\eta \in C_0^\infty$ and thus, for $1 \leqq p \leqq \infty$,

$$\|[(\tilde{u} - \hat{u})\eta]^{\vee}\|_p \leqq C\|F_m(\cdot, u)\|_p. \tag{3.7}$$

It follows from Lemma 3 and (3.7) that

$$\|[(\tilde{u} - \hat{u})\eta]^{\vee}\|_p \leqq C \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{3.8}$$

Combining (3.3), (3.6), and (3.8), we obtain (3.1) for $k = 0$. For $k = 1, \ldots, m$, apply the previous method to $\{D^\alpha[\eta\tilde{u} - \hat{u}]^{\vee}\}^{\wedge} = \xi^\alpha(\eta\tilde{u} - \hat{u})$. This leads to the linear functional $D^\alpha F_m(x, \cdot)$ for which Lemmas 2 and 3 are valid.

The definition of $\tilde{u}$ for u in H_m^p, $m > N/p$, may now be given. For u in H_m^p there exists a sequence $\{\varphi_j\}$ of C_0^∞ functions such that $\varphi_j \to u$ in H_m^p as $j \to \infty$. $\{\varphi_j\}$ is Cauchy in H_m^p and it follows from Lemma 4 that $\{(\eta_h\tilde{\varphi}_j)^{\vee}\}$ is a Cauchy sequence in H_m^p. Since the Fourier transform is continuous on the space of tempered distributions, $\eta_h\tilde{\varphi}_j$ has limit as $j \to \infty$, a tempered distribution with support Q_h. Define $\eta_h\tilde{u} = \lim \eta_h\tilde{\Phi}_j$ on Q_h and extend it by periodicity to all of R^N. It is easy to see that $\lim_{j\to\infty} \eta_h \tilde{\varphi}_j$ is independent of the choice of the sequence $\{\varphi_j\}$, so that $\tilde{u}$ is well defined. Note that $\|[\eta_h(\tilde{u} - \tilde{\varphi}_j)]^{\vee}\|_{p,m} \to 0$ as $j \to \infty$.

THEOREM 5. *Let* $1 \leqq p \leqq \infty$ *and* $m > N/p$. *There is a constant* C *such that for* $0 \leqq |\beta| \leqq m$, *for* $0 < h \leqq 1$, *and for all* u *in* H^p_m,

$$\|(\eta_h \tilde{u})^{\vee} - u\|_{p,k} \leqq Ch^{m-k} \sum_{|\alpha|=m} \|D^\alpha u\|_p.$$

Proof. Since C_0^∞ is dense in H^p_m, the result follows from Lemma 4.

The next result follows trivially from the Hausdorff-Young Theorem and from Theorem 5.

COROLLARY A. *Let* $1 \leqq p \leqq 2$ *and* $m > N/p$. *There is a constant* C *such that for* $0 \leqq |\beta| \leqq m$, *for* $0 < h \leqq 1$, *and for all* u *in* H^p_m,

$$\|\xi^\beta(\hat{u} - \eta_h \tilde{u})\|_{p'} \leqq Ch^{m-|\beta|} \sum_{|\alpha|=m} \|D^\alpha u\|_p,$$

where $1/p + 1/p' = 1$.

Finally, we state a consequence of the Poisson summation formula which will be of use in the next section.

THEOREM 6. *Let* $N = 1$ *and* $m \geqq 1$. *Then*

$$(\psi^{*m})^{\sim}(\xi) = \sum_{\mu \in Z} (\psi^{*m})^{\wedge}(\xi + 2\pi\mu).$$

4. Splines in R^N. In this section, we shall apply the techniques and results of Section 3 to a particular class of spline functions. Let $\psi = h^{-N}\chi_{2\pi/h}$ and $\Delta = Z_h^N$. If v is a function on Δ, then the function

$$s(x) = h^N \sum_{y \in \Delta} \psi^{*k+1}(x - y)v(y)$$

is said to be a spline of degree k. Regarded as a function of x, a spline of degree k is piecewise a polynomial of degree k and is C^{k-1}. Now, let u be a function on Δ. We say that $s(x)$ is a spline interpolant of order k for u provided there exists a function v on Δ such that

$$s(x) = h^N \sum_{y \in \Delta} \psi^{*k}(x - y)v(y)$$

and

$$s(x) = u(x) \quad \text{for all } x \in \Delta.$$

Let l^p denote the space $l^p(\Delta)$ with the usual discrete norm for $1 \leqq p \leqq \infty$. We shall establish the following inversion property.

THEOREM 7. *Let* $1 \leqq p \leqq \infty$, $u \in l^p$, *and let* k *be a positive integer. Then there exists a unique* $v \in l^p$ *such that*

$$Su(x) = h^N \sum_{y \in \Delta} \psi^{*k}(x - y)v(y)$$

is a spline interpolant of order k *for* u.

We shall prove the next lemma and then use it to prove the theorem.

LEMMA 8. *For* $k = 1, 2, \ldots, 1/(\psi^{*k})^{\sim}$ *has absolutely convergent Fourier series.*

Proof. Clearly, it suffices to consider $h = 1$. Since $(\psi^{*k})^{\sim}$ is the product of N functions, each of a single variable, it suffices to consider $N = 1$. Using Theorem 6 and (2.0), we have

$$(\psi^{*k})^{\sim}(\xi) = \sum_{l=-\infty}^{\infty} (\psi^{*k})^{\wedge}(\xi + 2\pi l)$$
$$= (\psi^{*k})^{\wedge}(\xi) + (2\pi)^{-1/2}(2 \sin \xi/2)^k$$
$$\cdot \sum_{l=1}^{\infty} (-1)^{lk}[(\xi + 2\pi l)^{-k} + (\xi - 2\pi l)^{-k}].$$

Thus we have $(\psi^{*k})^{\sim}(\xi) \geqq (\psi^{*k})^{\wedge}(\xi) \geqq (2\pi)^{-1/2}(2/\pi)^k$ for $-\pi < \xi \leqq \pi$. Since $(\psi^{*k})^{\sim}$ is periodic, we have $(\psi^{*k})^{\sim} \geqq C_k > 0$ everywhere. Obviously, $(\psi^{*k})^{\sim}$ has absolutely convergent Fourier series. The result now follows immediately from the Wiener-Levy theorem [3].

We turn now to the proof of Theorem 7. Again, it suffices to consider $h = 1$. Let a_μ, $\mu \in \Delta$, be the Fourier coefficients of $1/(\psi^{*k})^{\sim}$. Define v on Δ by

$$v(\mu) = \sum_{\nu \in \Delta} a_{\mu-\nu} u(\nu).$$

It follows from Lemma 8 that $v \in l^p$. Then the function $Su(x)$ defined in the theorem is obviously a spline interpolant of order k for u. The uniqueness of v is clear.

We shall now show that $Su(x)$ may be obtained directly from u.

THEOREM 9. *Let $u \in C_0^\infty$, $k \geqq 1$, and let Su be the spline interpolant of order k for u. Then*

$$Su(x) = \{[(\psi^{*k})^{\wedge}/(\psi^{*k})^{\sim}]\tilde{u}\}^{\vee}. \tag{4.1}$$

Proof. It suffices to consider $h = 1$. By Theorem 7, there exists $v \in l^p$ such that

$$Su(x) = \sum_{y \in \Delta} \psi^{*k}(x - y)v(y).$$

Hence

$$(Su)^{\wedge} = (2\pi)^{N/2}(\psi^{*k})^{\wedge}\tilde{v} \quad \text{and} \quad \tilde{u} = (Su)^{\sim} = (2\pi)^{N/2}(\psi^{*k})^{\sim}\tilde{v}.$$

It follows from Lemma 8 that

$$(Su)^{\wedge} = [(\psi^{*k})^{\wedge}/(\psi^{*k})^{\sim}]\tilde{u}.$$

Taking the inverse transform, we obtain (4.1).

Now we prove the following error estimate.

THEOREM 10. *Let $1 \leqq p \leqq \infty$ and $m > N/p$. Let $u \in C_0^\infty$ and let Su be the spline interpolant of order m for u. There is a constant C independent of u such that, for $0 \leqq j \leqq m$ and for $0 < h \leqq 1$, we have*

$$\|u - Su\|_{p,j} \leqq Ch^{m-j} \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{4.2}$$

Proof. It suffices to consider $h = 1$. We shall first prove (4.2) for $j = 0$. It follows from Theorem 9 that

$$\begin{aligned}(Su)^{\wedge} - \hat{u} &= (\psi^{*m})^{\sim -1}\{(\psi^{*m})^{\wedge}(\tilde{u} - \hat{u}) + [(\psi^{*m})^{\wedge} - (\psi^{*m})^{\sim}]\hat{u}\} \\ &= (\psi^{*m})^{\sim -1}\{(2\pi)^{-N/2}[F_m(\cdot, u)]^{\wedge} + [(\psi^{*m})^{\wedge} - (\psi^{*m})^{\sim}]\hat{u}\}.\end{aligned} \tag{4.3}$$

It follows from Lemma 8 and Lemma 3 that

$$\|\{(\psi^{*m})^{\sim -1}[F_m(\cdot, u)]^\wedge\}^\vee\|_p \leqq C \sum_{|\alpha|=m} \|D^\alpha u\|_p. \tag{4.4}$$

Now, consider the second term on the right in (4.3). We have

$$(\psi^{*m})^{\sim -1}[(\psi^{*m})^\sim - (\psi^{*m})^\wedge]\hat{u} = I_1 + I_2 \tag{4.5}$$

where

$$I_1 = (\psi^{*m})^{\sim -1}[(\psi^{*m})^\sim - (\psi^{*m})^\wedge]\eta\hat{u} \quad \text{and} \quad I_2 = (\psi^{*m})^{\sim -1}[(\psi^{*m})^\sim - (\psi^{*m})^\wedge](1-\eta)\hat{u}.$$

Since $\psi^{*m} \in L^1$ and $(\psi^{*m})^\sim$ is the Fourier transform of a bounded measure, it follows from Lemma 8 that

$$\|\check{I}_2\|_p \leqq C\|[(1-\eta)\hat{u}]^\vee\|_p.$$

Using (3.6), we obtain

$$\|\check{I}_2\|_p \leqq C \sum_{k=1}^N \|D_k^m u\|_p. \tag{4.6}$$

In order to estimate $\|\check{I}_1\|_p$, we shall apply (3.4) to $(\psi^{*m})^\sim - (\psi^{*m})^\wedge$. For $j = 1, \ldots, N$, define

$$\begin{aligned} \psi_j(x_j) &= 1, \quad -1 < 2x_j \leqq 1, \\ &= 0, \quad \text{otherwise} \end{aligned}$$

so that $\psi(x) = \prod_{j=1}^N \psi_j(x_j)$. Then, with $R = \{1, \ldots, N\}$ and J and K disjoint sets such that $J \cup K = R$ and $K \neq \varnothing$, we have

$$I_1 = (\psi^{*m})^{\sim -1}\eta \sum_{J,K} \left[\prod_{j\in J} (\psi_j^{*m})^\wedge\right] \prod_{k\in K} [(\psi_k^{*m})^\sim - (\psi_k^{*m})^\wedge]\hat{u}.$$

Writing $K = L \cup \{l\}$, we have

$$\begin{aligned} I_1 = (\psi^{*m})^{\sim -1}\eta \sum \left[\prod_{j\in J} (\psi_j^{*m})^\wedge\right]\left\{\prod_{k\in L} [(\psi_k^{*m})^\sim - (\psi_k^{*m})^\wedge]\right\} \\ \cdot \{[(\psi_l^{*m})^\sim - (\psi_l^{*m})^\wedge]/\xi_l^m\}(D_l^m u)^\wedge \end{aligned}$$

It follows from Theorem 6 that $a[(\psi_l^{*m})^\sim - (\psi_l^{*m})^\wedge]/\xi_l^m$ is in C_0^∞. Thus,

$$\|\check{I}_1\|_p \leqq C \sum_{k=1}^N \|D_k^m u\|_p. \tag{4.7}$$

For $j = 0$, (4.2) follows from (4.3), (4.4), (4.5), (4.6), and (4.7). For $j = 1, \ldots, m$, the previous steps are applied to $D^\alpha(Su - u)$ for $|\alpha| \leqq j$.

Remark. Let the hypotheses of Theorem 10 hold. Then, clearly, there is a constant C such that, for $u \in C_0^\infty$ and $0 < h \leqq 1$,

$$\|Su\|_{p,m} \leqq C\|u\|_{p,m}.$$

Thus, the definition of Su may be extended to all $u \in H_m^p$. We shall still call Su the spline interpolant of order m for u.

THEOREM 11. *Let* $1 \leqq p \leqq \infty$ *and let* $m > N/p$. *Let* $u \in H_m^p$ *and let* Su *be the spline interpolant of order* m *for* u. *There is a constant* C *independent of* u *such that, for* $0 \leqq j \leqq m$ *and for* $0 < h \leqq 1$,

$$\|u - Su\|_{p,j} \leqq Ch^{m-j} \sum_{|\alpha|=m} \|D^\alpha u\|_p.$$

Proof. Since C_0^∞ is dense in H_m^p, the result follows immediately from Theorem 10 and the previous remark.

COROLLARY B. *Let* $1 \leqq p \leqq 2$ *and* $m > N/p$. *Let* $u \in H_m^p$ *and let* Su *be the spline interpolant of order* m *for* u. *There is a constant* C *independent of* u *such that, for* $0 \leqq |\beta| \leqq m$ *and* $0 < h \leqq 1$, *we have*

$$\|\xi^\beta[\hat{u} - (Su)\hat{}\,]\|_{p'} \leqq Ch^{m-|\beta|} \sum_{|\alpha|=m} \|D^\alpha u\|_p.$$

5. The Case $1 < p < \infty$. In this section, we shall show that the error estimates in Theorems 5 and 11 need involve only pure derivatives when $1 < p < \infty$.

Let I be the set of indices τ such that $|\tau| = m$, $\tau_j = m$, and $\tau_l = 0$ for $l \neq j$, for $j = 1, \ldots, N$. Let P_I be the subset $\{r\}$ of P_m such that $D^\tau r = 0$ for all $\tau \in I$. Bramble and Hilbert [2] have proved the following result. ρ = diameter of Ω.

THEOREM 12. *Let* $1 < p < \infty$ *and let* F *be a linear functional on* $H_m^p(\Omega)$ *such that*

(i) $|F(u)| \leqq C \sum_{|\alpha| \leqq m} \rho^{|\alpha|-N/p} \|D^\alpha u\|_{p,\Omega}$ *for all* $u \in H_m^p(\Omega)$, *and*

(ii) $F(r) = 0$ *for all* $r \in P_I$.

Then there exists a constant C_1 *independent of* ρ *such that, for all* $u \in H_m^p(\Omega)$,

$$|F(u)| \leqq C_1 \rho^{m-N/p} \sum_{\tau \in I} \|D^\tau u\|_{p,\Omega}.$$

We shall apply this theorem with F the linear functional $F_m(x, \cdot)$. The proof of Lemma 2 shows that $F_m(x, r) = 0$ for all r in P_I. It follows from (2.3) that condition (i) of Theorem 12 is satisfied with $Q = \mu + Q_s$, $s = 2\pi/(m+1)$, $x \in \mu + Q_{2\pi}$.

It is possible to show, using Hörmander's multiplier theorem [4], that for $1 < p < \infty$ there exists a positive constant $C_{p,m}$ such that, for all $u \in H_m^p$,

$$C_{p,m}\|u\|_{p,m} \leqq \|u\|_p + \sum_{j=1}^N \|D_j^m u\|_p \leqq \|u\|_{p,m}. \tag{5.1}$$

We are now ready to state the improved error estimates for $1 < p < \infty$.

THEOREM 5′. *Let* $m > N/p$. *There is a constant* C *such that for* u *in* H_m^p, *for* $0 \leqq k \leqq m$, *and for* $0 < h \leqq 1$,

$$\|(\eta_h \tilde{u})\check{} - u\|_{p,k} \leqq Ch^{m-k} \sum_{j=1}^N \|D_j^m u\|_p. \tag{5.2}$$

Proof. It suffices to prove (5.2) for $u \in C_0^\infty$ and $h = 1$. With $k = 0$, we proceed as in the proof of Theorem 5 and obtain

$$\|(\eta\tilde{u})\check{} - u\|_p \leqq C\|F_m(\cdot, u)\|_p + C \sum_{j=1}^N \|D_j^m u\|_p. \tag{5.3}$$

It follows from Theorem 12 that

$$\|F_m(\cdot, u)\|_p \leqq C \sum_{j=1}^N \|D_j^m u\|_p. \tag{5.4}$$

(5.2) now follows from (5.3) and (5.4) for $k = 0$. For $k = 1, \ldots, m$, it follows from (5.1) and the case already proved that it is sufficient to show that

$$\sum_{j=1}^{N} \|D_j^k[(\eta\tilde{u})^{\vee} - u]\|_p \leqq C \sum_{j=1}^{N} \|D_j^m u\|_p.$$

This inequality follows from Theorem 12 and the proof of Lemma 4.

THEOREM 11′. *Let $m > N/p$. There is a constant C such that for u in H_m^p, for $0 \leqq k \leqq m$, and for $0 < h \leqq 1$,*

$$\|u - Su\|_{p,k} \leqq Ch^{m-k} \sum_{j=1}^{N} \|D_j^m u\|_p.$$

Proof. In view of Theorem 12 and (5.1), the result is established in a manner similar to that of the proof of Theorem 11.

Finally, we note that since χ_h is a Fourier multiplier with norm independent of h for $1 < p < \infty$, Theorem 5 and Corollary A for $1 < p < \infty$ and Theorem 5′ are valid with η_h replaced by χ_h.

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