

On the Eigenvectors of a Finite-Difference Approximation to the Sturm-Liouville Eigenvalue Problem

By Eckart Gekeler

Abstract. This paper is concerned with a centered finite-difference approximation to the nonselfadjoint Sturm-Liouville eigenvalue problem

$$L[u] = - [a(x)u_x]_x - b(x)u_x + c(x)u = \lambda u, \quad 0 < x < 1, \\ u(0) = u(1) = 0.$$

It is shown that the eigenvectors W_p of the $M \times M$ -matrix ($\Delta x = 1/(M + 1)$ mesh size), which approximates L , are bounded in the maximum norm independent of M if they are normalized so that $\|W_p\|_2 = 1$.

1. Introduction. The present paper is concerned with the nonselfadjoint problem

$$(1) \quad \begin{aligned} - [a(x)u_x]_x - b(x)u_x + c(x)u &= \lambda u, \quad 0 < x < 1, \\ u(0) &= u(1) = 0, \end{aligned}$$

where $a(x) \geq \alpha > 0$, $c(x) \geq 0$, and a, b, c are all bounded and smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ and a corresponding sequence of smooth eigenfunctions u^1, u^2, u^3, \dots (see, for instance Protter-Weinberger [10, p. 37] and Coddington-Levinson [4, p. 212]). Following Courant-Hilbert [5, p. 334], the eigenfunctions u^p are uniformly bounded in the supremum norm if they are normalized so that

$$\int_0^1 |u^p(x)|^2 dx = 1, \quad p = 1, 2, 3, \dots$$

Of course, by the well-known transformation

$$(2) \quad u(x) = \exp\left(-\frac{1}{2} \int_0^x \frac{b(t)}{a(t)} dt\right) w(x)$$

(1) may be put in the selfadjoint form

Received October 4, 1973.

AMS (MOS) subject classifications (1970). Primary 65L15.

Key words and phrases. Sturm-Liouville eigenvalue problem, finite-difference approximation.

Copyright © 1974, American Mathematical Society

$$- [a(x)w_x]_x + \tilde{c}(x)w = \lambda w, \quad 0 < x < 1,$$

$$w(0) = w(1) = 0,$$

where

$$\tilde{c}(x) = c(x) + \frac{1}{2}b_x(x) + \frac{1}{4}b^2(x)/a(x).$$

Here, in order to obtain $\tilde{c}(x) \geq 0$, we have to make a restricting assumption on b_x . Therefore, we choose the direct approximation of (1) by means of the finite-difference equations

$$(3) \quad -\frac{a_{k+1/2}(v_{k+1} - v_k) - a_{k-1/2}(v_k - v_{k-1})}{\Delta x^2} - b_k \frac{v_{k+1} - v_{k-1}}{2\Delta x} + c_k v_k = \Lambda v_k, \quad k = 1, \dots, M,$$

$$v_0 = v_{M+1} = 0,$$

where $M \in \mathbb{N}$, $\Delta x = 1/(M+1)$, and $v_k = v(k\Delta x)$. Equivalently, we may write (3) in matrix-vector notation

$$(3') \quad LV = \Lambda V,$$

where $V = (v_1, \dots, v_M)^T$ and the matrix L may be easily derived from (3).

Let $|b(x)| \leq \beta$ and $0 < \Delta x < 2\alpha/\beta$. Then the matrix L is equivalent to a real symmetric matrix (see Carasso [2]). Using this fact and Theorem 1.8 of Varga [11], it can be shown that all eigenvalues Λ_p of (3) are real and positive, $0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \dots \leq \Lambda_M$, and there exists a complete sequence of corresponding eigenvectors V^p . A result of Carasso [2, Corollary 1] says that there exist a constant K and an integer p_0 , both independent of M , such that

$$|V^p|_\infty = \max_{1 \leq k \leq M} |v_k^p| \leq Kp^{1/2}, \quad p_0 \leq p \leq M,$$

if

$$|V^p|_2^2 = \Delta x \sum_{k=1}^M |v_k^p|^2 = 1.$$

In the selfadjoint case, this result goes back to Bückner [1]. In this paper, we prove the following theorem:

THEOREM. *Let $a(x) \geq \alpha > 0$ and $c(x) \geq 0$, $0 < x < 1$. Assume that a , b , and c are differentiable bounded functions with bounded derivatives. Let $|b(x)| \leq \beta$. Let $0 < \Delta x \leq \alpha/\beta$ and let $\{V^p\}_{p=1}^M$ be the eigenvectors of (3), normalized so that $|V^p|_2 = 1$. Then*

$$|V^p|_\infty \leq \kappa, \quad p = 1, \dots, M,$$

for some constant κ independent of M .

Remark 1. In the case of the equation $u_{xx} = \lambda u$, this result may be proved by explicit computation of the eigenvectors V^P (see Isaacson-Keller [9, 9.1.1]).

Applications of the Theorem to the theory of finite-difference approximations to parabolic and hyperbolic partial differential equations are given in [3], [7], [8].

2. Proof of the Theorem. Instead of L , we consider, as in [2], the eigenvectors $D^{-1}V^P$ of the similar matrix $D^{-1}LD$ defined below. But, in contrast to Carasso [2], who uses a discrete maximum principle for his estimation, we then transpose the proof of Courant-Hilbert [5, p. 334] to the resulting discrete problem.

The following basic results are needed.

LEMMA 1 (CARASSO [2, LEMMA 1], [3, LEMMA 3.1]). Let $D = (d_1, \dots, d_M)$ be the diagonal matrix with

$$d_1 = 1, \quad d_i = + \left[\prod_{k=1}^{i-1} \frac{a_{k+1/2} - b_{k+1} \Delta x/2}{a_{k+1/2} + b_k \Delta x/2} \right]^{1/2}, \quad i = 2, \dots, M.$$

For $0 < \Delta x < 2\alpha/\beta$, we have $d_i > 0$ and

$$|D|_\infty \leq \kappa_1, \quad |D^{-1}|_\infty \leq \kappa_2$$

for constants κ_1, κ_2 independent of M . Furthermore, $D^{-1}LD = (P + Q)/\Delta x^2$ where $P = (p_{ik})_{i,k=1,\dots,M}$,

$$\begin{aligned} p_{ik} &= (p_{k+1/2} + p_{k-1/2}), & i = k, \\ &= -p_{k+1/2}, & i = k + 1, \\ &= -p_{i+1/2}, & k = i + 1, \\ &= 0, & \text{otherwise,} \end{aligned}$$

$$p_{k+1/2} = (a_{k+1/2} - b_{k+1} \Delta x/2)^{1/2} (a_{k+1/2} + b_k \Delta x/2)^{1/2},$$

and $Q = (q_1, \dots, q_M)$ is the diagonal matrix with

$$q_k = (a_{k+1/2} + a_{k-1/2}) - (p_{k+1/2} + p_{k-1/2}) + \Delta x^2 c_k.$$

Remark 2. The change of variables $V = DW$ is a discrete analog to (2) [2].

LEMMA 2 (CARASSO [2, THEOREM 1]). Let Λ_p, V^P be the characteristic pairs of the matrix L with $|V^P|_2 = 1$. Let u^P be an eigenfunction of (1) corresponding to λ_p and let U^P be the vector of dimension M obtained from u^P by mesh-point evaluation. Assume u^P normalized so that $|D^{-1}U^P|_2 = |D^{-1}V^P|_2$, then, as $\Delta x \rightarrow 0$, we have

$$(4) \quad |\lambda_p - \Lambda_p| \leq \kappa_3(p) \Delta x^2, \quad |U^P - V^P|_2 \leq \kappa_4(p) \Delta x^2,$$

where κ_3, κ_4 are positive constants depending only on p .

In the selfadjoint case, Lemma 2 was proved by Gary [6].

Remark 3. The estimation (4) implies $|U^p - V^p|_\infty \leq \kappa_4(p)\Delta x^{3/2}$.

LEMMA 3. *Let*

$$C_l(W) = \sum_{k=1}^l [p_{k+1/2}(w_k - w_{k+1}) - p_{k-1/2}(w_k - w_{k-1})] q_k w_k / \Delta x^2.$$

Then, under the assumptions of the Theorem,

$$C_l(W^p) = -q_l p_{l+1/2} w_{l+1}^p w_l^p / \Delta x^2 + O(1), \quad l = 1, \dots, M,$$

where $W^p = D^{-1}V^p$ and $O(1)$ denotes a function which has a bound independent of M .

Proof. We show at first that $|q_k / \Delta x^2| \leq \kappa_5$ independently of M . To this end, it suffices to consider $a_{k+1/2} - p_{k+1/2}$. By means of the binomial theorem, we obtain

$$\begin{aligned} a_{k+1/2} - p_{k+1/2} \\ = a_{k+1/2} - a_{k+1/2} \left(1 - \frac{b_{k+1}\Delta x}{4a_{k+1/2}} + O(\Delta x^2) \right) \left(1 + \frac{b_k\Delta x}{4a_{k+1/2}} + O(\Delta x^2) \right) \end{aligned}$$

Inserting $b_{k+1} = b_k + O(\Delta x)$, we find that

$$(5) \quad a_{k+1/2} - p_{k+1/2} = O(\Delta x^2).$$

Now, since $w_0^p = 0$,

$$\begin{aligned} C_l(W^p) &= -q_l p_{l+1/2} w_{l+1}^p w_l^p / \Delta x^2 \\ &+ \sum_{k=1}^l \frac{q_k}{\Delta x^2} (p_{k+1/2} - p_{k-1/2}) w_k^p w_k^p + \sum_{k=1}^{l-1} \frac{q_{k+1} - q_k}{\Delta x^2} p_{k+1/2} w_{k+1}^p w_k^p. \end{aligned}$$

But, by the mean value theorem, we have $p_{k+1} - p_k = O(\Delta x)$ and $(q_{k+1} - q_k) / \Delta x^2 = O(\Delta x)$. Hence, using Schwarz's inequality and $|W^p|_2 \leq \kappa_6$, we obtain the desired result.

Now, according to Lemma 1, it suffices to prove the Theorem for the eigenvectors $W^p = D^{-1}V^p$ of the matrix $(P + Q) / \Delta x^2$ which also has the eigenvalues Λ_p . We multiply the k th row of

$$(1/\Delta x^2)(P + Q)W^p = \Lambda_p W^p$$

by

$$p_{k+1/2}(w_k^p - w_{k+1}^p) - p_{k-1/2}(w_k^p - w_{k-1}^p)$$

and obtain, by adding all rows from $k = 1$ to $k = l$,

$$(6) \quad \left[\frac{p_{l+1/2}(w_l^p - w_{l+1}^p)}{\Delta x} \right]^2 + C_l(W^p) + \Lambda_p p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x \sum_{k=1}^l p_x(\xi_k) w_k^p w_k^p = \left[\frac{p_{1/2}(w_1^p - w_0^p)}{\Delta x} \right]^2,$$

where $(k - 1/2)\Delta x < \xi_k < (k + 1/2)\Delta x$. In order to eliminate the term on the right side of (6), we sum up Eqs. (6) for $l = 1$ to $l = M$, add $(p_{1/2}(w_1^p - w_0^p)/\Delta x)^2$ to both sides, and divide by $M + 1 = 1/\Delta x$. Then

$$(7) \quad \left[\frac{p_{1/2}(w_1^p - w_0^p)}{\Delta x} \right]^2 = \Delta x \sum_{l=0}^M \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x} \right]^2 + \Delta x \sum_{l=1}^M C_l(W^p) + \Lambda_p \Delta x \sum_{l=1}^M p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x^2 \sum_{l=1}^M \sum_{k=1}^l p_x(\xi_k) w_k^p w_k^p.$$

But

$$(8) \quad 0 < \alpha/2 < p_{l+1/2} < \kappa_7 \quad (\kappa_7 > 1)$$

if $0 < \Delta x \leq \alpha/\beta$. Thus, using the fundamental relation

$$\sum_{l=0}^M p_{l+1/2}(w_{l+1} - w_l)^2 = W^T P W$$

($w_0 = w_{M+1} = 0$), we derive

$$\begin{aligned} \Delta x \sum_{l=0}^M \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x} \right]^2 &\leq \Delta x \kappa_7 (W^p)^T P W^p / \Delta x^2 \\ &= \Delta x \kappa_7 (W^p)^T (P + Q) W^p / \Delta x^2 - \Delta x \kappa_7 \sum_{l=1}^M q_l w_l^p w_l^p / \Delta x^2 \\ &\leq \Delta x \kappa_7 \Lambda_p (W^p)^T W^p + \kappa_5 \kappa_7 \Delta x (W^p)^T W^p = \kappa_7 \Lambda_p + \kappa_5 \kappa_7, \end{aligned}$$

since $|q_l/\Delta x^2| \leq \kappa_5$ independently of M and $|W^p|_2^2 = 1$. Hence, applying Schwarz's inequality, we find from (7) by means of the assumption and Lemma 3 that

$$[p_{1/2}(w_1^p - w_0^p)/\Delta x]^2 \leq \kappa_8 \Lambda_p + \kappa_9.$$

From this estimation, Eq. (6), and Lemma 3, we deduce that

$$(\Lambda_p - q_l/\Delta x^2)p_{l+1/2}w_{l+1}^pw_l^p \leq \kappa_{10}\Lambda_p + \kappa_{11}.$$

Consequently, observing (8), we obtain, in case $\Lambda_p > \kappa_5$, that

$$(9) \quad w_{l+1}^pw_l^p \leq \kappa_{12}, \quad l = 1, \dots, M,$$

for some constant κ_{12} independent of M . For $\Lambda_p \leq \kappa_5$, the assertion of the Theorem follows by Lemma 2, Remark 3.

Finally, we return once more to Eq. (6). The above estimation yields

$$\left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x} \right]^2 \leq \kappa_{13}\Lambda_p + \kappa_{14}, \quad l = 1, \dots, M,$$

or, using (9),

$$\begin{aligned} |w_l^p|^2 &\leq |w_l^p|^2 + |w_{l+1}^p|^2 = (w_{l+1}^p - w_l^p)^2 + 2w_l^pw_{l+1}^p \\ &= \frac{\Delta x^2}{p_{l+1/2}^2} \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x} \right]^2 + 2w_{l+1}^pw_l^p \\ &\leq \Delta x^2(\kappa_{15}\Lambda_p + \kappa_{16}) + \kappa_{17} = \kappa, \quad l = 1, \dots, M. \end{aligned}$$

Hence, $\max_{1 \leq p \leq M} \{|w^p|_\infty\}$ is bounded independently of M .

Mathematics Department
Universität Stuttgart
7000 Stuttgart, Germany

1. H. BÜCKNER, "Über Konvergenzsätze, die sich bei der Anwendung eines Differenzenverfahrens auf ein Sturm-Liouvillesches Eigenwertproblem ergeben," *Math. Z.*, v. 51, 1944, pp. 423-465. MR 11, 58.
2. A. CARASSO, "Finite-difference methods and the eigenvalue problem for nonselfadjoint Sturm-Liouville operators," *Math. Comp.*, v. 23, 1969, pp. 717-729. MR 41 #2938.
3. A. CARASSO & S. V. PARTER, "An analysis of 'boundary-value techniques' for parabolic problems," *Math. Comp.*, v. 24, 1970, pp. 315-340. MR 44 #1249.
4. E. A. CODDINGTON & N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955. MR 16, 1022.
5. R. COURANT & D. HILBERT, *Methoden der Mathematischen Physik*. Vol. 1, Interscience, New York, 1953. MR 16, 426.
6. J. GARY, "Computing eigenvalues of ordinary differential equations by finite differences," *Math. Comp.*, v. 19, 1965, pp. 365-379. MR 31 #4163.
7. E. GEKELER, "Long-range and periodic solutions of parabolic problems," *Math. Z.*, v. 134, 1973, pp. 53-66.
8. E. GEKELER, "A-convergence of finite-difference approximations of parabolic initial boundary value problems," *SIAM J. Numer. Anal.* (To appear.)

9. E. ISAACSON & H. B. KELLER, *Analysis of Numerical Methods*, Wiley, New York, 1966.
MR 34 #924.
10. M. H. PROTTER & H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967. MR 36 #2935.
11. R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
MR 28 #1725.