

Chebyshev Approximation by Interpolating Rationals on $[\alpha, \infty)$

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Abstract. Decay-type functions with a finite number of zeros are approximated on $[\alpha, \infty)$ by an oscillation factor times a negative power of a polynomial. Best approximations are characterized and an algorithm indicated.

Consider the problem of approximation on $[\alpha, \infty)$ of a continuous function f decaying to zero at ∞ . Let us assume, as does Williams [4], that f can be represented as $f(x) = B(x)g(x)$, where B is a polynomial and g is a positive continuous function on $[\alpha, \infty)$. We consider approximation of g by a negative power of a positive polynomial.

Let $C[\alpha, \infty)$ be the set of continuous functions on $[\alpha, \infty)$ which vanish at ∞ . For $h \in C[\alpha, \infty)$, define

$$\|h\| = \sup \{|h(x)|: \alpha \leq x < \infty\}.$$

Let B be a polynomial of degree m . Let g be an element of $C[\alpha, \infty)$ such that $B(x)g(x) \in C[\alpha, \infty)$ and $g > 0$ for $\alpha \leq x < \infty$. Let $p > 0$ and $(n-1)p > m$. Define

$$L(A, x) = \sum_{k=1}^n a_k x^{k-1}, \quad G(A, x) = 1/[L(A, x)]^p.$$

Let $P = \{A: L(A, x) > 0 \text{ for } \alpha \leq x \leq \infty, a_n \neq 0\}$. For $A \in P$, $B(x)G(A, x) \rightarrow 0$ as $x \rightarrow \infty$. The approximation problem is to find A^* to minimize

$$e(A) = \|Bg - BG(A, \cdot)\|$$

over $A \in P$. Such a parameter A^* is called best. Virtually the same problem was raised by Williams [4] for approximation on a finite interval.

It should be noted that the parameter space P is an open subset of n -space, which is a requirement of many theories, in particular that of [2]. If the restriction $a_n \neq 0$ were dropped, P would no longer be open: for example, 1 is positive on $[\alpha, \infty)$, but $1 - x/k$ is not.

LEMMA 1. Let $A, C \in P$ and $G(A, \cdot) - G(C, \cdot)$ have n zeros on $[\alpha, \infty)$; then $A = C$.

Proof.

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$$G(A, x) - G(C, x) = [L(C, x)]^p - [L(A, x)]^p / [L(A, x)L(C, x)]^p,$$

which has n zeros only if $L(A, x) - L(C, x)$ vanishes at the same points, which implies $A = C$.

Define for $A \in P$

$$D(A, C, x) = \sum_{k=1}^n c_k \frac{\partial}{\partial a_k} G(A, x);$$

then

$$D(A, C, x) = -pL(C, x)/[L(A, x)]^{p+1}, \quad D(A, C, \infty) = 0,$$

giving

LEMMA 2. $\{D(A, C, \cdot) : C \in E_n\}$ is a Haar subspace of dimension n on $[\alpha, \infty)$ for $A \in P$.

THEOREM. Let $B = 1$. A is best if and only if $g - G(A, \cdot)$ alternates n times. A best parameter is unique.

The theorem follows immediately from Theorems 4 and 5 of [2] and change of variable.

The general approximation problem is one of Chebyshev approximation of g by G with respect to the nonnegative (and possibly unbounded) multiplicative weight function $|B|$.

THEOREM. A is best if and only if $|B|(g - G(A, \cdot))$ alternates n times. A best parameter is unique.

Proof. Sufficiency follows from Lemma 1 and arguments similar to those of Lemma 2 of [1]. Suppose $|B|(g - G(A, \cdot))$ alternates less than n times. As ∞ is a zero of the error curve, there is \bar{x} such that

$$B(x)(g(x) - G(A, x)) < \epsilon(A)/2, \quad x \geq \bar{x}.$$

Assertion. There is $\epsilon > 0$ such that if $|c_k - a_k| < \epsilon$, $k = 1, \dots, n$, then

$$B(x)(g(x) - G(C, x)) < 3\epsilon(A)/4, \quad x \geq \bar{x}.$$

By classical arguments, in particular those of Williams, there is C arbitrarily close to A such that the maximum error of C on $[\alpha, \bar{x}]$ is less than the maximum error of A on $[\alpha, \bar{x}]$. Combining this with the assertion we have necessity of alternation. Alternation and arguments similar to those of Lemma 2 of [1] give uniqueness.

The theorems are similar to the characterization result of Williams [4, p. 201]. The alternation result suggests using the algorithm of Williams or adapting the algorithm of the author for transformed polynomial approximation [3] to compute best approximations. In the case $p = 1$, the approximation problem is one of weighted approximation by the reciprocal of a polynomial of degree n , that is, weighted

approximation by $R_n^0[\alpha, \infty)$. Best approximations may be computed by the standard rational Remez algorithm.

Existence of best approximations is not guaranteed in the approximation problem of this paper. However, recent work of Taylor and Williams [6] shows that existence is not guaranteed in the problem of Williams [4] either.

In the preceding analysis, the approximations were restricted and, in particular, constant $L(A, \cdot)$ was not permitted. What happens in general with fewer restrictions on approximations, is not known. However, the case where $B = 1$ has been fully analyzed by Brink [5], who showed that if we only require that $L(A, \cdot)$ be positive, we do not get a standard alternating theory.

A generalization of the problem of this paper is to have $\{\phi_1, \dots, \phi_n\}$ a sequence such that

- (i) it is a sequence of increasing powers,
- (ii) the highest power r satisfies $rp > m$,
- (iii) it is a Chebyshev set on $[\alpha, \infty)$.

We then define

$$L(A, x) = \sum_{k=1}^n a_k \phi_k(x).$$

The coefficient of $\phi_n(x) = x^r$ is a_n . It is readily verified that the same theory holds. Similar algorithms can be used.

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