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# Discrete Rational $L_p$ Approximation

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**Abstract.** In this paper, the problem of approximating a function defined on a finite subset of the real line by a family of generalized rational functions whose numerator and denominator spaces satisfy the Haar conditions on some closed interval $[a, b]$ containing the finite set is considered. The pointwise closure of the family restricted to the finite set is explicitly determined. The representation obtained is used to analyze the convergence of best approximations on discrete subsets of $[a, b]$ to best approximations over the whole interval (as the discrete subsets become dense) in the case that the function approximated is continuous on $[a, b]$ and the rational family consists of quotients of algebraic polynomials. It is found that the convergence is uniform over $[a, b]$ if the function approximated is a so-called normal point. Only $L_p$ norms with $1 \leqslant p < \infty$ are employed.

**Introduction.** In the application of nonlinear approximation theory one is normally constrained to the calculation of best approximations on finite subsets of the underlying domain. Moreover, the discrete problem may be more difficult than the continuous problem, in the sense that the discretized family may not be (pointwise) closed so that best approximations need not exist. In this paper, these problems will be investigated for a family of generalized rational functions defined on a closed subinterval of the real line.

In particular, let $P$ and $Q$ be Haar subspaces of $C[a, b]$ of dimension $n$ and $m$, respectively, let $X = \{x_1, \cdots, x_M\} \subset [a, b]$ with $M \geqslant m + n + 1$ and let $R(X) \equiv \{p/q \mid p \in P,\ q \in Q$ and $q(x) \neq 0$ for all $x \in X\}$. Then, if $\|\cdot\|$ is a norm on $B(X) \equiv \{f \mid f$ is a real-valued function on $X\}$ and $f \in B(X)$ is given, we seek an $r^* \in R(X)$ such that $\|f - r^*\| = \inf_{r \in R(X)} \|f - r\|$. The case when $\|f\| = \max_{x \in X} |f(x)|$ has been extensively studied (see [1] and [2] for example) and we shall consider only norms of the form $\|f\|_t = [\Sigma_{x \in X} |f(x)|^t]^{1/t}$, where $1 \leqslant t < \infty$.

The techniques that will be used are somewhat different from those useful for uniform approximation since there is no characterization theorem available for these norms.

*Remark* 1. The family $R(X)$ is usually defined by requiring that, if $p/q \in R(X)$, then $q(x) > 0$ for all $x \in X$. However, there seems little to be gained by this requirement since it does not preclude the existence of a pole in $[a, b]$, even at a best approximation, and there is usually no simple way to maintain this condition during computation. Moreover, as will be seen, for sufficiently dense discrete subsets using

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ordinary rational approximation and least-square approximation, say, the condition $q(x) > 0$ for all $x \in X$ will hold for a best approximation anyway.

**Existence of Discrete Best Approximations.** We begin by identifying explicitly the pointwise closure $\overline{R(X)}$ of the set $R(X)$ in $B(X)$. It is clear that $\overline{R(X)}$ is an existence set in the sense that each $f \in B(X)$ has a closest point in $\overline{R(X)}$ with respect to any norm on $B(X)$. The notation $R(Y)$, where $Y$ is some subset of $[a, b]$, will denote the set $\{p/q \mid p \in P,\ q \in Q \text{ and } q(x) \neq 0 \text{ for all } x \in Y\}$.

The following example illustrates the "exceptional" types of elements that may appear in $\overline{R(X)}$.

*Example* 1. Let $X = \{-3, -2, -1, 0, 1, 2, 3\}$ and

$$R(X) = R_2^2(X) = \left\{ \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2} \,\middle|\, b_0 + b_1 x + b_2 x^2 \neq 0 \ \text{ for all } \ x \in X \right\}.$$

(a) Let $p_v(x) = x^2 + 1/v$ and $q_v(x) = x^2 - 1/v$, $v = 2, 3, \cdots$. Then $p_v(x)/q_v(x) \longrightarrow 1$ at all $x$ except $x = 0$ where $p_v(0)/q_v(0) = -1$. The limiting function is clearly not an element of $R(X)$ itself.

(b) Let $p_v(x) = 1/v$ and $q_v(x) = x^2 - (1 - 1/v)^2$. Then $p_v(x)/q_v(x) \longrightarrow 0$ except at $x = \pm 1$ where the limiting value is ½. Again, the limiting function is not in $R(X)$.

As will be seen, the types of behavior in (a) and (b) above are the only ones to be dealt with in describing $\overline{R(X)}$. To clarify the presentation of Theorem 1, we have the following definition.

*Definition.* Let $S_1$ denote the set of functions $g$ in $B(X)$ such that there exists some set $S \subset X$ (depending on $g$) containing at most $k = \min(n - 1, m - 1)$ elements and some rational function $p/q$ in $R(X \sim S)$ with $p(x) = q(x) = 0$ for all $x \in S$ for which $g = p/q$ on $X \sim S$.

Let $S_2$ denote the set of all functions $g$ in $B(X)$ such that $g$ is zero except precisely on some subset $T \subset X$ (depending on $g$) having at most $m - 1$ elements.

*Remark* 2. We should note here that the sets $S_1$ and $S_2$ are not disjoint since $\{0\} \subset S_1 \cap S_2$ and that $S_1 \supset R(X)$. In general, neither is a subset of the other. Also, some elements of $S_1$ may not have a unique pair $S$ and $p/q \in R(X \sim S)$ corresponding to them. For example, in $R_2^2(X)$, with $X$ as in Example 1, the function $g(x) = 1$ is "represented" by the empty set and $1/1$, but also by $S = \{0\}$ and $x/x \in R(X \sim \{0\})$, etc. This ambiguity will not affect what follows. In Example 1 above, the limiting function in (a) is in $S_1$. That is, the function equals $p/q \equiv x^2/x^2 \in R(X \sim \{0\})$, except on $S = \{0\}$ where its value is $-1$. Note also that $p = q = 0$ on $S$. Similarly, in (b) the limiting function is in $S_2$, since it vanishes except at precisely 2 ($= m - 1$) points where its value is ½.

THEOREM 1. *The set* $\overline{R(X)}$ *is given by* $S_1 \cup S_2$.

*Proof.* Suppose first that $\{r_v\} = \{p_v/q_v\}$ is a sequence in $R(X)$ converging pointwise on $X$ to some $g \in B(X)$. Let $N(\varphi) = \max_{x \in X} |\varphi(x)|$ for all $\varphi \in B(X)$. Clearly, we may assume $N(q_v) = 1$ for all $v$. Now, $\{N(r_v)\}$ is bounded and so, for some $0 \leqslant K < \infty$, we have $K > N(p_v/q_v) \geqslant N(p_v)$. Now, $N(\cdot)$ is a norm on both $P$ and $Q$, since both are Haar subspaces and $M \geqslant m + n + 1$, and so there exist subsequences (which we do not relabel) $\{p_v\}$ and $\{q_v\}$ and elements $p \in P$ and $q \in Q$ such that $p_v \longrightarrow p$ and $q_v \longrightarrow q$ uniformly on $[a, b]$ where $N(q) = 1$. Thus $p_v(x)/q_v(x) \longrightarrow p(x)/q(x)$ for all $x \in X$ except, perhaps, on $S \equiv \{x \in X | q(x) = 0\}$. Denoting the cardinality of a set $A$ by $C(A)$, we have that $C(S) \leqslant m - 1$ since $q$ is not identically zero and $Q$ is Haar of dimension $m$. Also, since $\{N(r_v)\}$ is bounded, we have that $p(x) = 0$ for all $x \in S$. We consider two cases:

(a) $p \not\equiv 0$. Then $C(S) \leqslant \min(m - 1, n - 1)$ and $g(x) = p(x)/q(x)$ for each $x \in X \sim S$ and $p/q \in R(X \sim S)$. Thus $g \in S_1$.

(b) $p \equiv 0$. Again $C(S) \leqslant m - 1$ and, clearly, $g(x) = 0$ on $X \sim S$. Letting $T$ be the subset of $S$ where $g$ is not zero, we conclude $g \in S_2$.

Thus, $g \in S_1 \cup S_2$ and so $\overline{R(X)} \subset S_1 \cup S_2$. To show the opposite inclusion, suppose $g \in S_1 \cup S_2$ and consider two cases.

(a) $g \in S_1$. Then $g(x) = p(x)/q(x) \in R(X \sim S)$ for $x \in X \sim S$ where $C(S) \leqslant \min(n - 1, m - 1) \equiv k$ and $p(x) = q(x) = 0$ on $S$. Pick $p_1 \in P$ and $q_1 \in Q$ such that $q_1(x) > 0$ for all $x \in X$ and $p_1(x) = g(x)q_1(x)$ for all $x \in S$. Both of these choices are possible by the elementary properties of Haar spaces and since $C(S) \leqslant k$. Define $r_\epsilon = (p + \epsilon p_1)/(q + \epsilon q_1)$. On $X \sim S$, $r_\epsilon(x) \longrightarrow p(x)/q(x) = g(x)$ as $\epsilon \longrightarrow 0$ and on $S$, $r_\epsilon(x) = p_1(x)/q_1(x) = g(x)$ so that $r_\epsilon \longrightarrow g$ on $X$ and hence $g \in \overline{R(X)}$.

(b) $g \in S_2$. Then $g(x)$ is nonzero exactly on $T \subset X$ where $C(T) \leqslant m - 1$. Pick $p_1 \in P$ such that $p_1(x) > 0$ for all $x \in X$ and $q_1 \in Q$ such that $q_1(x) = p_1(x)/g(x)$ for all $x \in T$. Finally, let $q \in Q$ be such that $q(x) = 0$ for all $x \in T$ and $q(x) \neq 0$ for every $x \in X \sim T$ (e.g., let $q$ have $(m - 1) - C(T)$ roots between two consecutive points of $X$, be nonzero at some point of $X \sim T$ and be zero on $T$. Since this requires interpolation at $m$ points, this is always possible.) Then, for $\epsilon$ sufficiently small and positive, let $r_\epsilon = \epsilon p_1/(q + \epsilon q_1)$. Then, $r_\epsilon(x) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ for all $x \in X \sim T$ and $r_\epsilon(x) = g(x)$ for $x \in T$ and so $g \in \overline{R(X)}$. Thus, $S_1 \cup S_2 \subset \overline{R(X)}$ and the proof is complete. □

COROLLARY 1. *Let*

$$R(X) \equiv R_m^n(X) \equiv \left\{ \frac{a_0 + \cdots + a_n x^n}{b_0 + \cdots + b_m x^m} \,\middle|\, b_0 + \cdots + b_m x^m \neq 0 \text{ for all } x \in X \right\}.$$

*Then* $\overline{R_m^n(X)} = S_1 \cup S_2$ *where* $S_1$ *can be written as*

$$S_1 = \left\{ g \in B(X) \mid g(x) = \frac{p^*(x)}{q^*(x)} \text{ on } X \sim S, \text{ where } \frac{p^*}{q^*} \in R^{n-l}_{m-l}(X \sim S) \right.$$
$$\left. \text{for some } S \subset X \text{ with } C(S) = l \leqslant k = \min(m, n) \right\}$$

*and $S_2$ is as before.*

*Proof.* This follows immediately from Theorem 1 by cancelling the common roots of $p$ and $q$ on $S$ in the statement of Theorem 1. □

To further illustrate Corollary 1, we refer back to (a) of Example 1. There the limiting function $g(x) \in S_1$ was equal to $x^2/x^2$ except on $S = \{0\}$ where its value was $-1$. After cancelling the $l = 2$ common roots of $p(x) = x^2 = q(x)$ on $S = \{0\}$, we have $p^*(x) = 1 = q^*(x)$ and $p^*/q^* \in R^0_0(X \sim \{0\})$. In general, it is possible that $q^*$ still has roots in $S$, so that the statement $p^*/q^* \in R^{n-l}_{m-l}(X \sim S)$ cannot be strengthened.

A natural question to ask is: When does a function in $B(X)$ have a best approximation in $\overline{R(X)}$ that is not in $R(X)$? Necessary conditions for this occurrence are contained in the following result.

Corollary 2. *Let $1 \leqslant t < \infty$ be arbitrary. Then, if $f \in B(X)$ has $g \in S_1$ (or $g \in S_2$) as a best approximation, $g = f$ on the associated set $S(T)$. In the case $R(X) \equiv R^n_m(X)$, if $g \in S_1$ is a best approximation to $f$, then the function $p/q \in R^{n-l}_{m-l}(X \sim S)$ where $g = p/q$ on $X \sim S$ is a best approximation to $f$ (restricted to $X \sim S$) from $R^{n-l}_{m-l}(X \sim S)$ where $l = C(S)$.*

*Proof.* If $g$ is a best approximation to $f$ from $\overline{R(X)}$ and disagrees with $f$ at some point of $S$ (or $T$ if $g \in S_2$), then redefinition of $g(x)$ at this point to be equal to $f(x)$ gives an element of $\overline{R(X)}$ that is clearly a better approximation to $f$, thus yielding a contradiction. To obtain the second conclusion, assume it is false and find $p_1/q_1 \in R^{n-l}_{m-l}(X \sim S)$ that is a better approximation than $p/q$. But then, defining $h(x)$ to be $p_1/q_1$ on $X \sim S$ and $f(x)$ on $S$ and applying Corollary 1, we obtain a better approximation to $f$ than $g$—a contradiction. □

The following example shows that a function not in $\overline{R(X)}$ may have a best approximation that is not in $R(X)$, so that the situation described in Corollary 2 can occur.

*Example* 2. Let $X = \{-2, -1, 0, 1, 2\}$ and consider approximation by elements of $R(X) = R^1_1(X)$ with respect to the norm $\|\varphi\| = \Sigma_{x \in X} |\varphi(x)|$ for all $\varphi \in B(X)$. Define the function $f$ by $f(\pm 1) = 9$, $f(\pm 2) = -1$ and $f(0) = 3$. Then the function $g \in \overline{R(X)}$ which is 3 at $x = 0$ and vanishes elsewhere satisfies $\|g - f\|_1 = 1$. Clearly, no constant function does better than $g$ over $X$ and if $r(x) = (a + bx)/(c + dx)$ with $ad - bc \neq 0$, and $c + dx \neq 0$ for $x \in X$, then a simple check using the strict monotonicity of $r$ shows that $\|f - r\|_1 > 1$. By Corollary 1, the only remaining elements of $\overline{R(X)}$ are constant on $X$ except for exactly one point. It is clear from the graph of $f$ that, using constant functions where one can ignore one point of $X$, it is impossible to do better than $g$. Finally, defining $g$ to have any constant value between 0 and $-1$ on $X \sim \{0\}$ and the value 3 at $x = 0$ yields the same error. Thus, $f$ has, in fact, infinitely many best approximations from $\overline{R(X)}$ none of which are in $R(X)$.

Another simple application of Theorem 1 is the following.

COROLLARY 3. *Let $f$ be a strictly positive continuous function on $[a, b]$ such that its maximum value is less than twice its minimum value. Then $f$ has a best approximation in $R_1^0(X)$ for any finite subset $X$ of $[a, b]$ containing at least three points.*

*Proof.* Let $f$ be any function satisfying the hypotheses of the theorem. The only elements of $\overline{R_1^0(X)}$ that are not in $R_1^0(X)$ are those that are identically zero except at exactly one point. Clearly, if such a function $g$ were a best approximation, $g$ would agree with $f$ at some point where $f$ had its maximum, say at $x_1$. Let $x_2$ be any point where $f$ achieves its minimum over $[a, b]$ and consider the constant function $r(x) = f(x_1)$ for all $x \in X$. Then, for any $x \neq x_1$,

$$\begin{aligned} |r(x) - f(x)| &= |f(x_1) - f(x)| \leqslant |f(x_1) - f(x_2)| = f(x_1) - f(x_2) < f(x_2) \\ &= |f(x_2) - g(x)| \leqslant |f(x) - g(x)|, \end{aligned}$$

so that $g$ cannot be a best approximation. Whence the best approximation must lie in $R_1^0(X)$. □

**Convergence of Discrete Approximations.** We consider here the following specific problem. Given $f$ in $C[0, 1]$, it is desired to calculate a best approximation to $f$ from $R_m^n[0, 1] = \{p/q \,|\, p \in P_n, q \in P_m$ and $q(x) > 0$ for all $x \in [0, 1]\}$ (where $P_k$ denotes the polynomials of degree $k$ or less) with respect to some $L_t$ norm with $1 \leqslant t < \infty$. To accomplish this, $[0, 1]$ is replaced by a sequence of grids of the form $[h_v] = \{k/N_v \,|\, k = 0, 1, \cdots, N_v\}$ where $N_v \longrightarrow \infty$, and the $L_t$ norm is replaced by its discrete analog

$$\|g\|_{[h_v]} = h_v^{1/t}\left(\sum_{x\in[h_v]} |g(x)|^t\right)^{1/t},$$

where the endpoints have been included for notational convenience. Then a best approximation $g_v \in \overline{R_m^n([h_v])}$ is calculated for each $v$. The natural and important question is: Does the sequence $\{g_v\}$ converge (in some sense) to a best approximation $r$ in $R_m^n[0, 1]$?

*Remark* 3. The choice of equally spaced grids is for convenience only. In general, one uses the discrete norm

$$\left[\sum_{j=0}^{N-1} |g(x_j)|^t (x_{j+1} - x_j)\right]^{1/t};$$

the results obtained below are still valid once it is noted that the results of the preceding section could have been obtained using weighted discrete norms of the form

$$\|g\| = \left[\sum_{j=0}^{N} |g(x_j)|^{t} w_j\right]^{1/t} \qquad \text{where } w_j \geqslant 0, j = 0, \cdots, N.$$

The following concept of normality is basic to the whole problem of convergence.

*Definition.* An element $r \in R_m^n[0, 1]$ is called normal if $r \in R_m^n[0, 1] \sim R_{m-1}^{n-1}[0, 1]$. Also, for $1 \leqslant t \leqslant \infty$ arbitrary but fixed, define $NP$ as $\{f \in C[0, 1] \mid f$ has only normal best approximations in $R_m^n[0, 1]\}$.

*Remark* 4. By a theorem of Cheney and Goldstein [3], for $1 < t < \infty$, $NP = C[0, 1] \sim R_{m-1}^{n-1}[0, 1]$ if $m \geqslant 1$. If $t = 1$ (or $\infty$), an element $f$ not in $R_m^n[0, 1]$ may have a nonnormal best approximation [4].

In what follows, the symbol $\|g\|_A$, where $A$ is a subset of $[h_v]$ and $g \in B([h_v])$, will denote $h_v^{1/t}(\Sigma_{x \in A} |g(x)|^t)^{1/t}$. To simplify notation, we will shorten $\|\cdot\|_{[h_v]}$ to $\|\cdot\|_v$. The $L_t$ norm on $[0, 1]$ will be denoted by $\|\cdot\|_t$.

LEMMA 1. *Let $t$ be arbitrary with $1 \leqslant t < \infty$. For each $v$, let $[h_v]$ denote the set $\{k/N_v \mid k = 0, \cdots, N_v\}$ where $\{N_v\}$ is a sequence of positive integers such that $N_v \longrightarrow \infty$. Let $f \in NP$ and select for each $v$ a best approximation to $f$ from $R_m^n([h_v])$ with respect to the corresponding discrete $L_t$ norm. Then, there exists a $v_0$ such that for all $v \geqslant v_0$, $g_v$ is not in the set $S_2$ of Theorem* 1.

*Proof.* Assume the lemma is false. Then, there is a sequence of subsets of $[h_v]$, say $\{T_v\}$, such that each $T_v$ contains at most $m$ elements and such that $g_v = f$ on $T_v$ and vanishes elsewhere. Let $A_v \equiv [h_v] \sim T_v$ and let $r_0$ denote a best approximation to $f$ over $[0, 1]$ from $R_m^n[0, 1]$. Then $\|f - g_v\|_v \leqslant \|f - r_0\|_v \longrightarrow \|f - r_0\|_t$ as $v \longrightarrow \infty$ and $\|f - g_v\|_v = \|f\|_{A_v} \longrightarrow \|f\|_t$ as $v \longrightarrow \infty$, since each $T_v$ has at most $m$ points and $f$ is continuous. Thus,

$$\|f\|_t = \overline{\lim_v}\, \|f - g_v\|_v \leqslant \overline{\lim_v}\, \|f - r_0\|_v = \|f - r_0\|_t\,,$$

which is a contradiction since $f \in NP$ and 0 is not a normal element of $R_m^n[0, 1]$. □

LEMMA 2. *Let $t$ be arbitrary in $[1, \infty)$, $f$ continuous on $[0, 1]$ and $m \geqslant 1$. Assume that, for each $v$, $g_v \in S_1$ is a best approximation to $f$ with respect to the norm $\|\cdot\|_v$ and let $r_v \equiv p_v/q_v \in R_{m-l_v}^{n-l_v}([h_v] \sim S_v)$ be such that $g_v = p_v/q_v$ on $[h_v] \sim S_v$ where $S_v$ is some subset of $[h_v]$ containing $l_v \leqslant \min(m, n)$ elements. Then, there is a set $F \subset [0, 1]$ whose complement has Lebesgue measure zero and an element $r$ of $R_{m-l}^{n-l}[0, 1]$ where $l = \underline{\lim}\, l_v$ such that, for some subsequence $\{r_{v_j}\}$, we have $r_{v_j}(x) \longrightarrow r(x)$ for all $x \in F$. In fact, $F = \bigcup_{j=1}^{\infty} F_j$ where each $F_j$ is a finite union of closed subintervals with $F_j \subset F_{j+1}$ such that $r_{v_k} \longrightarrow r$ uniformly on each $F_j$.*

*Proof.* Let $A_v = [h_v] \sim S_v$. The sequence $\{\|r_v\|_{A_v}\}$ is bounded since $\|f - r_v\|_{A_v} \leqslant \|f - g_v\|_v \leqslant \|f\|_v \longrightarrow \|f\|_t$ as $v \longrightarrow \infty$. Moreover, we may assume $\|q_v\|_\infty = 1$ for all $v$.

*Claim.* $\{\|p_v\|_\infty\}$ is bounded.

*Proof.* Assume this to be false and let $r'_v = r_v/\|p_v\|_\infty$. By passing to a subsequence if necessary, we may assume that $\|r'_v\|_{A_v} \longrightarrow 0$. Let $p'_v$ denote $p_v/\|p_v\|_\infty$. Then we may assume that $p'_v \longrightarrow p^*$ and $q_v \longrightarrow q^*$ uniformly where $\|p^*\|_\infty = \|q^*\|_\infty = 1$ and where $p^*$ and $q^*$ are polynomials of degree at most $n$ and $m$, respectively. Then, $r'_v \longrightarrow p^*/q^*$ uniformly on each closed subset of the set $\{x \mid q^*(x) \neq 0\}$. Pick a closed subinterval $I \equiv [\alpha, \beta] \subset [0, 1]$ such that neither $p^*$ nor $q^*$ has a root in $I$ and let $B_v$ denote the set $([h_v] \sim S_v) \cap I$. Then, $\inf_{x \in I} |p^*(x)| \equiv \delta > 0$ so that

$$\|r'_v\|_{B_v} \geqslant \left\| \frac{p'_v}{\|q_v\|_\infty} \right\|_{B_v} = \|p'_v\|_{B_v} = \left(\frac{1}{N_v}\right)^{1/t} \left[\sum_{x \in B_v} |p'_v(x)|^t\right]^{1/t}$$
$$\geqslant \left(\frac{\theta_v}{N_v}\right)^{1/t} \left(\frac{\delta}{2}\right)$$

for $v$ sufficiently large, where $\theta_v$ is the number of elements in $B_v$. But $\theta_v/N_v \longrightarrow \beta - \alpha$ as $v \longrightarrow \infty$. Hence, for $v$ sufficiently large, $\|r'_v\|_{A_v} \geqslant \|r'_v\|_{B_v} \geqslant (\beta - \alpha)^{1/t}\delta/4 > 0$, which is a contradiction and so the claim is proved.

Thus, $\{\|p_v\|_\infty\}$ is bounded and so there exist subsequences (which we do not relabel) $\{p_v\}$ and $\{q_v\}$ and polynomials $p$ and $q$ such that $p_v \longrightarrow p$ and $q_v \longrightarrow q$ uniformly where $\|q\|_\infty = 1$. As above, $r_v \equiv p_v/q_v$ converges to $r \equiv p/q$ uniformly on closed subsets of $\{x \in [0, 1] \mid q(x) \neq 0\}$. Now, this set can be written as $\bigcup_{j=1}^\infty F_j$, where each $F_j$ is a finite union of closed intervals with $F_j \subset F_{j+1}$ for all $j$ and where $r_v \longrightarrow r$ uniformly on each $F_j$. Letting $A_{vj} = A_v \cap F_j$, we have $\|r\|_{A_{vj}} \leqslant \|r_v - r\|_{A_{vj}} + \|r_v\|_{A_{vj}}$ so that

$$\overline{\lim_v} \|r\|_{A_{vj}} \leqslant \overline{\lim_v} \|r_v\|_{A_{vj}} + 0 \leqslant M < \infty,$$

where $M$ is some constant independent of $j$. But

$$\overline{\lim_v} \|r\|_{A_{vj}} = \left[\int_{F_j} |r(x)|^t dx\right]^{1/t}$$

for each $j$ since $r$ is continuous on $F_j$ and so $\int_{F_j} |r(x)|^t dx \leqslant M^t$ for all $j$. Since $F = \bigcup_{j=1}^\infty F_j$ has measure one and since $F_j \subset F_{j+1}$, we conclude from the monotone convergence theorem that $r \in L_t[0, 1]$. But this means that $r$ has at worst removable singularities in $[0, 1]$. Thus, $r \in R_{m-k}^{n-k}[0, 1]$ for some $k \geqslant l = \underline{\lim}\, l_v$ and the proof is complete. □

We are now ready for the following theorem which is the main result of this paper.

THEOREM 2. *Let $t$ be arbitrary in $[1, \infty)$ and let $f$ be a continuous function on $[0, 1]$ that has only normal best approximations in $R_m^n[0, 1]$. Then there exists a $v_0$ such that, for all $v \geqslant v_0$, $f$ has a best approximation in $R_m^n([h_v])$. Moreover, if*

*$\{r_v\}$ is any sequence of best approximations to $f$ from $R^n_m([h_v])$, then every subsequence of $\{r_v\}$ has a further subsequence uniformly convergent to some best approximation to $f$ from $R^n_m[0, 1]$.*

*Proof.* By Lemma 1, we may assume that any best approximation to $f$ from $\overline{R^n_m([h_v])}$ is not in $S_2$. Suppose that there exists a sequence of best approximations $\{g_v\} \subset S_1$ where $g_v = p_v/q_v \in R^{n-l_v}_{m-l_v}([h_v] \sim S_v)$ on $[h_v] \sim S_v$ where $S_v$ has $l_v$ elements with $\min(m, n) \geqslant l_v \geqslant l_0 > 0$. We wish to show that the assumption $l_v \geqslant l_0 > 0$ leads to a contradiction.

Let $r$ be any best approximation to $f$ from $R^n_m[0, 1]$. Then $\|f - g_v\|_v \leqslant \|f - r\|_v \longrightarrow \|f - r\|_t$ as $v \longrightarrow \infty$ so that $\overline{\lim}_v \|f - g_v\|_v \leqslant \|f - r\|_t$. By Corollary 2,

$$\operatorname{dist}(f, R^{n-l_v}_{m-l_v}([h_v] \sim S_v)) = \|f - g_v\|_v.$$

Thus,

$$\overline{\lim_v} \operatorname{dist}(f, R^{n-l_v}_{m-l_v}([h_v] \sim S_v)) \leqslant \operatorname{dist}(f, R^n_m[0, 1]).$$

By Lemma 2, there is an $r' \in R^{n-l_0}_{m-l_0}[0, 1]$ and a subsequence (which we do not relabel) $\{r_v\}$ such that $r_v \longrightarrow r'$ uniformly on each closed subset of a set $F = \bigcup_{j=1}^{\infty} F_j$ whose complement has measure zero. Let $\epsilon > 0$ be given and pick $k$ so that $\|f - r'\|_{A_v} \leqslant \|f - r'\|_{A_{vk}} + \epsilon$ where $A_v = [h_v] \sim S_v$ and $A_{vk} = A_v \cap F_k$. Then $\|f - r'\|_{A_v} \leqslant \|f - r_v\|_{A_{vk}} + \epsilon + \|r' - r_v\|_{A_{vk}}$ and thus

$$\begin{aligned}\|f - r'\|_t = \overline{\lim_v} \|f - r'\|_{A_v} &\leqslant \epsilon + \overline{\lim_v} \|f - r_v\|_{A_v} + 0 \\ &\leqslant \epsilon + \operatorname{dist}(f, R^n_m[0, 1]).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\|f - r'\|_t \leqslant \operatorname{dist}(f, R^n_m[0, 1])$ which is a contradiction, since $f \in NP$ but $r' \in R^{n-l_0}_{m-l_0}[0, 1]$ is not normal.

Thus, no subsequence of $\{l_v\}$ is bounded away from zero and, since each $l_v$ is an integer, eventually $l_v = 0$. Hence, $S_v$ is eventually empty and hence, for all $v$ greater than some $v_0$, each best approximation is in $R^n_m([h_v])$.

The remainder of the theorem follows from Lemma 2 (and its proof) and the above argument which shows that any cluster point (in the sense of Lemma 2) of a sequence of best approximations $\{r_v\}$ is a best approximation to $f$ from $R^n_m[0, 1]$ and hence is normal and can have no singularities in $[0, 1]$. Thus, the corresponding subsequence of $\{r_v\}$ will eventually have no singularities in $[0, 1]$ and the convergence will thus be uniform on the entire interval. □

The following is immediate from Theorem 2.

COROLLARY 4. *If $f \in NP$ has a unique best approximation $r$ in $R^n_m[0, 1]$, then any sequence $\{r_v\}$ of best approximations from $R^n_m([h_v])$, $v = 1, 2, \cdots$, converges uniformly over $[0, 1]$ to $r_0$.*

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