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Differential-Difference Properties of Hypergeometric Polynomials*

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Abstract. We develop differential-difference properties of a class of hypergeometric polynomials which are a generalization of the Jacobi polynomials. The formulas are analogous to known formulas for the classical orthogonal polynomials.

I. Introduction. In this paper, we derive a differential-difference equation satisfied by the hypergeometric polynomials

$$(1)\qquad P_n(x) = {}_{p+2}F_q\left(\begin{matrix}-n,\ n+\lambda,\ a_1,\ a_2,\cdots,\ a_p\\ b_1,\ b_2,\cdots,\ b_q\end{matrix}\middle|\, x\right),\qquad n=0,1,2,\cdots.$$

Throughout, we employ the shorthand notation

$$(2)\qquad (a_p+n) = \prod_{j=1}^{p}(a_j+n), \text{ etc.,}$$

see [1]. In general, where any variable is subscripted by a p or q, it is to be understood that the shorthand notation has been invoked.

II. Results.

THEOREM. *Let*

(i) $\lambda \neq 1, 2, \cdots$;

(ii) *none of the quantities* b_j, λ, $\lambda + 1 - b_j$ *be negative integers or zero*, $j = 1, 2, \cdots, q$;

(iii) *no* $b_j =$ *any* a_h, $h = 1, 2, \cdots, p$; $j = 1, 2, \cdots, q$. *Then the polynomials* $P_n(x)$ *satisfy the differential-difference equation*

$$(3)\qquad (\epsilon x - \delta x^2)\frac{dP_n(x)}{dx} = \sum_{\nu=0}^{\sigma}(A_\nu + xB_\nu)P_{n-\nu}(x),$$

where

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$$\delta = \begin{cases} 1, & p+1 \geqslant q, \\ 0, & p+1 < q, \end{cases} \qquad \epsilon = \begin{cases} 0, & p+1 > q, \\ 1, & p+1 \leqslant q, \end{cases} \qquad \sigma = \max\{p+1, q\}, \tag{4}$$

and no such equation of lower order $\sigma' < \sigma$ exists. The A_ν's and B_ν's are unique and

$$A_\nu = \begin{cases} (-n)_\nu [(1-n-\lambda)_\nu]^{-1} (2\nu - 2n - \lambda) \\ \quad \cdot \left\{ (-)^{\nu+1}\epsilon + \dfrac{(-)^{\sigma+1}}{\nu!} \displaystyle\sum_{s=0}^{\nu} \dfrac{(-\nu)_s (n-s)(b_q + n - s - 1)}{(\nu + s - 2n - \lambda)_{\sigma+1-\nu}} \right\}, & \nu > 0; \\ n \left\{ \epsilon - \dfrac{(b_q + n - 1)}{(2n + \lambda - \sigma)_\sigma} \right\}, & \nu = 0; \end{cases} \tag{5}$$

$$B_\nu = \begin{cases} (-n)_\nu [(1-n-\lambda)_\nu]^{-1} (2\nu - 2n - \lambda) \\ \quad \cdot \left\{ (-)^{\nu}\delta + \dfrac{(-)^{\sigma+1}}{\Gamma(\nu)} \displaystyle\sum_{s=0}^{\nu-1} \dfrac{(1-\nu)_s (a_p + n - s - 1)}{(\nu + s + 1 - \lambda - 2n)_{\sigma-\nu}} \right\}, & \nu > 0, \\ -\delta n, \quad \nu = 0. \end{cases} \tag{6}$$

Proof. By equating coefficients of x^{k+1} in (3) we find

$$\begin{aligned} &(k+1)\{\epsilon(k-n)(a_p+k)\beta_{-1}(k) - \delta k(b_q+k)\beta_0(k)\} \\ &\equiv (a_p+k)\sum_{\nu=0}^{\sigma} C_\nu \alpha_{\nu+1}(k)\beta_{\nu-1}(k) + (k+1)(b_q+k)\sum_{\nu=0}^{\sigma} D_\nu \alpha_\nu(k)\beta_\nu(k), \end{aligned} \tag{7}$$

where

$$\begin{aligned} &\begin{bmatrix} C_\nu \\ D_\nu \end{bmatrix} = \frac{(-)^\nu (1-n-\lambda)_\nu}{(-n)_\nu} \begin{bmatrix} A_\nu \\ B_\nu \end{bmatrix}, \\ &\qquad \alpha_\nu(k) = (k-n)_\nu, \qquad \beta_\nu(k) = (n+\lambda+k-\sigma)_{\sigma-\nu}. \end{aligned} \tag{8}$$

The above can be considered an identity between polynomials in the (generally complex-valued) variable k. If $p + 1 > q$, (7) requires that two polynomials of degree $p + \sigma + 2$ be identical; this condition furnishes $p + \sigma + 3$ equations in $2\sigma + 2$ unknowns, so that we must have $\sigma \leqslant p + 1$. If $p + 1 = q$, we similarly find $\sigma \leqslant p + 1$, while, if $p + 1 < q$, we find that $\sigma \leqslant q$. Thus

$$\sigma \leqslant \max\{p+1, q\}. \tag{9}$$

Now, if we assume equality above, the A_ν and B_ν (if they exist) are unique. Suppose there is another such recurrence relation with coefficients A_ν^* and B_ν^*. Subtracting these two, we have

$$(10) \qquad 0=\sum_{\nu=0}^{\sigma} [(A_\nu - A_\nu^*)+x(B_\nu - B_\nu^*)]P_{n-\nu}(x),$$

but this is impossible, under the hypotheses (ii) and (iii), since the author has shown that in this case any linear difference equation satisfied by $P_n(x)$ must be of order $\sigma + 1$, at least, see [1].

Now, if $q = p + 1$, (7) holds if and only if

$$(11) \qquad (k+1)[\beta_0(k+1)\epsilon - (b_q + k)] \equiv \sum_{\nu=0}^{\sigma} C_\nu \alpha_\nu(k+1)\beta_\nu(k+1),$$

$$(12) \qquad (n+\lambda+k-\sigma)[-\delta k\beta_1(k+1)+(k-n)(a_p+k)] \equiv \sum_{\nu=0}^{\sigma} D_\nu \alpha_\nu(k)\beta_\nu(k).$$

(Note that a suitable linear combination of (11) and (12) gives (7), i.e., multiply (11) by $(k - n)(n + \lambda + k - \sigma)(a_p + k)$ and (12) by $(k + 1)(b_q + k)$ and add.) To establish (11) for $p + 1 = q$, we observe that it represents an identity between two polynomials in k, each of degree $q + 2$ and each having two identical factors. It only remains to show that (11) holds for $q + 1$ distinct values of k. Assume that all the quantities $-1, -b_j, j = 1, 2, \cdots, q$, are distinct and let k have these values in (7). The result is (11) evaluated at these values.

Similarly to show (12) for $p + 1 = q$, we need only prove that it holds for the $p + 2$ values (assumed distinct) $n, \sigma - n - \lambda, -a_j, j = 1, 2, \cdots, p$. This is true, since (7) and (12) for these values are the same.

(The requirement that the values of k chosen above be distinct may be relaxed by continuity.)

Now, replace x by $x/a_j, j = p' + 1, p' + 2, \cdots, q - 1$ in (3), where $p' < q - 1$. This shows that

$$(13) \qquad \epsilon x\frac{dP_n'(x)}{dx} = \sum_{\nu=0}^{\sigma} (C_\nu + xD_\nu')P_{n-\nu}(x)(-)^\nu(1-n-\lambda)_\nu/(-n)_\nu,$$

where $P_n'(x)$ is $P_n(x)$ with p replaced by p' and

$$(14) \qquad D_\nu' = \lim_{a_u\to\infty}\ \lim_{a_{u+1}\to\infty} \cdots \lim_{a_v\to\infty} [D_\nu/a_u a_{u+1}\cdots a_v], \qquad u=p'+1,\ v=q-1.$$

The same limit process applied to (12) yields the following equation for the determination of D_ν':

$$(15) \qquad (k-n)(n+\lambda+k-\sigma)(a_{p'}+k) = \sum_{\nu=0}^{\sigma} D_\nu'\alpha_\nu(k)\beta_\nu(k).$$

The equation for C_ν in this case is (11) as it stands. But (11) and (15) together are (11) and (12), respectively, written for $p + 1 < q$.

Similarly, replacing x by xb_j, $j = q' + 1, q' + 2, \cdots, p + 1, q' \leqslant p$, and letting $b_j \longrightarrow \infty$ in (3) gives

$$-x^2 \frac{dP''_n(x)}{dx} = \sum_{\nu=0}^{\sigma} (C'_\nu + xD_\nu)P''_{n-\nu}(x)(-)^\nu(1 - n - \lambda)_\nu/(-n)_\nu, \tag{16}$$

where

$$C'_\nu = \lim_{b_u \to \infty} \lim_{b_{u+1} \to \infty} \cdots \lim_{b_v \to \infty} (C_\nu / b_u b_{u+1} \cdots b_v), \quad u = q' + 1, \ v = p + 1, \tag{17}$$

and $P''_n(x)$ is $P_n(x)$ with q replaced by q'. This limit process applied to (11) gives

$$-(k + 1)(b_q + k) = \sum_{\nu=0}^{\sigma} C'_\nu \alpha_\nu(k + 1)\beta_\nu(k + 1), \tag{18}$$

and (12) is used unchanged for D_ν. These two equations, though, are just (11) and (12) for $p + 1 > q$.

Thus (11) and (12) are established for all p, q and we have succeeded in "uncoupling" Eq. (7) to give Eqs. (11) and (12), which involve C_ν and D_ν alone, respectively.

Next, we solve these two equations.

In Eq. (11), let $k + 1 - n = -s$, $s = 0, 1, 2, \cdots, \sigma$. The result can be written

$$\sum_{\nu=0}^{s} \frac{(-)^\nu C_\nu (-s)_\nu}{(s + 1 - 2n - \lambda)_\nu} = \epsilon(n - s) + \frac{(-)^{\sigma+1}(n - s)(n + b_q - s - 1)}{(s + 1 - 2n - \lambda)_{\sigma - s}}, \tag{19}$$
$$s = 0, 1, 2, \cdots, \sigma.$$

But if $1 - 2n - \lambda \neq 0, -1, -2, \cdots$, the above equation can be solved for C_ν by applying a lemma of Wimp [1]. After some algebra and evaluation of ${}_2F_1$'s of unit argument, one arrives at (5). To find the D_ν's, let $k - n = -s$ in (12) and proceed in a similar fashion.

III. Concluding Remarks. If $p + 1 = q$ and $x = 1$ in (3), we get a recursion relation for $P_n(1)$ of order $\max(p + 1, q)$. Note that this is of order one less than that obtained by putting $x = 1$ in the homogeneous linear difference equation satisfied by $P_n(x)$ given in [1]. If $p = 1$, the resulting recursion relation for

$${}_3F_2\left(\begin{matrix} -n, n + \lambda, a_1 \\ b_1, b_2 \end{matrix} \middle| \, 1\right) \tag{20}$$

is that given by Bailey [3], which in turn is Watson's result [2] slightly rewritten. For $p + 1 = q$ and x general, (3) of course provides a generalization of the classical differential-difference formula for the Jacobi polynomials, see [4, p. 170 (15)].

A differential-difference relation for the polynomials

$$Q_n(x) = {}_{p+1}F_q\left(\begin{matrix} -n, a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| \, x\right) \tag{21}$$

can easily be obtained from (3) by replacing x by x/λ and letting $\lambda \longrightarrow \infty$.

We point out that the conditions of the theorem can be relaxed considerably. If λ is a positive integer m, we can write

$$(-n)_\nu/(1-n-\lambda)_\nu = n!(n+1-\nu)_{m-1}/\Gamma(n+m), \tag{22}$$

which is well defined for all n, so condition (i) is not essential to the analysis.

Also, if any of the quantities (ii) are negative integers or zero, limits may be taken after the equation has been multiplied by a suitable factor, see [1]. The quantity n can even be nonintegral when $q > p + 1$ or when $q = p + 1$ and $|\arg(1-x)| < \pi$, by the permanence principle for functional equations. (It may be necessary, in this case, to multiply the equation by a factor $(r - n - \lambda)$ to make the coefficients well defined.)

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