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A Partition Formula for the Integer Coefficients of the Theta Function Nome

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Abstract. In elliptic function theory, the nome q can be given as a power series in ϵ with integer coefficients, $q = \Sigma_{n\geqslant 0}\delta_n\epsilon^{4n+1}$. Heretofore, the first 14 coefficients were calculated with considerable difficulty. In this paper, an explicit and general formula involving partitions is given for all the δ_n. A table of the first 59 of these integers is given. The table is of number-theoretical interest as well as useful for calculating complete and incomplete elliptic integrals.

It has been pointed out almost everywhere [1]–[8] that the complete and incomplete elliptic integrals of the first and second kind can be calculated using the rapidly convergent theta functions. For example, the complete elliptic integral of the first kind, with modulus k, has the expressions, [5],

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{\pi}{2}\left(1 + 2\sum_{n\geqslant 1} q^{n^2}\right)^2 \tag{1}$$

$$= \frac{\pi}{2}\left(1 + 4\sum_{n\geqslant 1} \frac{q^n}{1 + q^{2n}}\right). \tag{2}$$

The nome q is a function of the modulus k via the intermediary ϵ,

$$\epsilon = \frac{1}{2}\left[\frac{1 - (k')^{1/2}}{1 + (k')^{1/2}}\right], \qquad k' = (1 - k^2)^{1/2}, \tag{3}$$

where (Weierstrass [6])

$$q = q(\epsilon) = \sum_{n\geqslant 0} \delta_n \epsilon^{4n+1}. \tag{4}$$

The first few integer coefficients, δ_n, $\delta_0 = 1$, $\delta_1 = 2$, $\delta_2 = 15$, $\delta_3 = 150$, $\delta_4 = 1707, \ldots$, have been calculated, the first four by Weierstrass [6], six by Milne-Thomson [7], fourteen by Lowan, et al. [8]. In this paper, we give a completely general formula for all of these integers δ_n and calculate the first 59. These match exactly with those of [6], [7], [8].

THEOREM. *Let the nome q, ϵ and δ_n, $n \geqslant 1$, be as defined above. Then the integers*

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TABLE 1

$$\epsilon(q) = \sum_{k \geq 0} C_k q^{4k+1}$$

C_k	k	C_k	k
1	0	-26556084	51
-2	1	32566736	52
5	2	-39865632	53
-10	3	48714496	54
18	4	-59425460	55
-32	5	72370451	56
55	6	-87991906	57
-90	7	106815482	58
144	8	-129465152	59
-226	9	156680498	60
346	10	-189337106	61
-522	11	228470640	62
777	12	-275304480	63
-1138	13	331282178	64
1648	14	-398105538	65
-2362	15	477778404	66
3348	16	-572657866	67
-4704	17	685514048	68
6554	18	-819598930	69
-9056	19	978726474	70
12425	20	-1167365856	71
-16932	21	1390748755	72
22922	22	-1654993826	73
-30848	23	1967251104	74
41282	24	-2335868000	75
-54946	25	2770581364	76
72768	26	-3282739616	77
-95914	27	3885557834	78
125842	28	-4594412154	79
-164402	29	5427179280	80
213901	30	-6404625958	81
-277204	31	7550857344	82
357904	32	-8893832810	83
-460448	33	10465956914	84
590330	34	-12304758144	85
-754368	35	14453668032	86
960948	36	-16962912538	87
-1220370	37	19890533666	88
1545306	38	-23303559744	89
-1951258	39	27279342357	90
2457152	40	-31907085620	91
-3086112	41	37289594562	92
3866271	42	-43545269554	93
-4831786	43	50810383492	94
6024144	44	-59241680384	95
-7493554	45	69019335360	96
9300676	46	-80350328386	97
-11518784	47	93472287408	98
14236130	48	-108657859786	99
-17558850	49	126219686290	100
21614512	50		

(5) $$\delta_n = \sum_{1 \leqslant k \leqslant n} (-1)^k \sum{}^{*} \frac{(4n+k)!}{(4n+1)!a_1! \ldots a_n!} C_1^{a_1} \ldots C_n^{a_n},$$

where Σ^* *is a summation over all the integer partitions* $k = \Sigma_{1 \leqslant s \leqslant n} a_s$, $n = \Sigma_{1 \leqslant s \leqslant n} s a_s$, $a_s \geqslant 0$. *The* C_k *are integers satisfying the recursion relation*

Table 2

$$q(\epsilon) = \sum_{n \geq 0} \delta_n \epsilon^{4n+1}$$

δ_n	n
1	0
2	1
15	2
150	3
1707	4
20910	5
268616	6
3567400	7
48555069	8
673458874	9
9481557398	10
135119529972	11
1944997539623	12
28235172753886	13
412850231439153	14
6074299605748746	15
89857589279037102	16
1335623521633805028	17
19936473955587624656	18
298710089390201812048	19
4490774010052865283744	20
67720348285795481957568	21
1024048736429572060655319	22
15524537851637363478060414	23
235895735639949246721659678	24
3592055029374068300374296756	25
54804356948531406780686077335	26
837668718981768081882232416270	27
12825003353647588436048753554935	28
196661581715188108128970795004550	29
3020038525632669791764002713422532	30
46440290652254605592799214895101504	31
715038427200862813001136930265147512	32
11022543326741896130999268335183824336	33
170106631417212663570292309396189783082	34
2627963818429706645211842908688374293300	35
40639684325725031991118847449107074979172	36
629057442741114268349966425119727665581336	37
9745798971459876443490275061762301063120623	38
1511116397169795848764748621445190045537049166	39
2345060743702465972443964863972690339335580295	40
36418974504781217231871247380322076634437243526	41
566000002327718606520661245108793975375803955227	42
8802486794330300502560080205926854680407227244446	43
136987427943746069763667154538164819216291647222299	44
2133189839056434593163452664751723335632221518780078	45
33238368972758722951581782375851915672575296334699144	46
518203119165380946481537680006811067385208271890196520	47
8083508347684372357034150759458280957643776575928292964	48
126162240900543525003409858218743000597507890709208990152	49
1970058936472178165658457810603802393218322368577430279357	50
30778014840073879593583180632414116424005881087226151493994	51
481066708903723582902283009313154592694417730034725440142805	52
7522557895177030497054105691715628210384957408473671745338098	53
117683064927025633752081377689230738120581991161302571300341950	54
1841805068163667424373785303397419166787012535392792974110785668	55
28836865945637549391397672719750764886865336155791435146300051614	56
451669686692109094028753360935093143169400258806504175758417989996	57
7077117970879174771151304161914657388532532100161812093166648499271	58

(6) $$C_k = \delta_k^{m(m+1)} - 2 \sum_{0<s\leqslant\sqrt{k}} C_{k-s^2},$$

where $C_0 = 1, C_1 = -2, C_2 = 5, C_3 = -10, C_4 = 18, \ldots,$ *and* $\delta_k^{m(m+1)}$ *is one if* k *is a product of consecutive integers,* $k = m(m + 1)$ *and zero otherwise.*

Proof. Weierstrass [6, p. 275] showed that

(7) $$\left(1 + 2 \sum_{k\geqslant 1} q^{4k^2}\right) \epsilon = \sum_{k\geqslant 1} q^{(2k-1)^2}.$$

After performing the power series division to solve for ϵ, we find

(8) $$\epsilon = \sum_{k\geqslant 0} C_k q^{4k+1},$$

where the integers C_k satisfy the recursion relation (5). All we have to do now is solve for q in Eq. (8). We do this by the technique of series reversion. For example, Jolley [9] and Van Orstand [10] give the first seven and twelve coefficients, respectively, for power series reversion. Finding the general term in this development is an old combinatorial problem first solved by McMahon [11]. Note that Eq. (8) as a power series has zero coefficients except for every fourth term. This fact simplifies McMahon's formula to Eq. (5).

Previous calculations of the δ_n [6], [7], [8] were done by rewriting (6) to give

(9) $$q = \epsilon + \sum_{k\geqslant 1} 2\epsilon q^{(2k)^2} - q^{(2k+1)^2}.$$

This expression can be used as a recursion relation to eliminate q on the right-hand side. It is clear from this that the δ_n are integers.

Table I gives the integers C_k, $k = 0, 1, 2, \ldots, 100$. The recursion relation defining them only requires $\sqrt{k}$ terms for each k, hence calculation of the C_k is not time consuming. However, constructing [13] Table II, δ_n, $n = 0, 1, 2, \ldots, 58$, involves generating all the partitions of n for each n. The number $p(n)$ of partitions of n is $O(n^{-1}e^{\alpha\sqrt{n}})$, Rademacher [12]. The decimal representation of δ_{58} has 67 digits, involved 715, 220 partitions, and took over four hours on an IBM 7030 [13]. The operations were performed by variable-length multiple-precision STRAP subroutines.

We thank the referee for pointing out that Rauch [14, Section 7], using his version of the above theorem, checked the coefficients of Lowan, et al. [8]. Computationally, the two techniques are identical, e.g., both involve sums over partitions.

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13. We thank the BYU Scientific Computation Center and their resurrected IBM 7030 (an old STRETCH) for doing the extensive number crunching of which Table II is a distillation.

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