

## Cubature Formulas of Degree Nine for Symmetric Planar Regions

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**Abstract.** A method of constructing 19-point cubature formulas with degree of exactness 9 is given for two-dimensional regions and weight functions which are symmetric in each variable. For some regions, e.g., the square and the circle, these formulas can be reduced to 18-point formulas.

**1. Introduction.** We consider cubature formulas,

$$(1) \quad \iint_R w(x, y) f(x, y) dx dy \cong \sum_{k=1}^N w_k f(x_k, y_k),$$

which are exact for all polynomials in  $x$  and  $y$  of degree  $\leq d$  but not for all polynomials of degree  $d + 1$ . Such formulas are said to have degree  $d$ .

According to Rabinowitz and Richter [1], we say that formula (1) is a 'good' formula if it has all of its points  $(x_k, y_k)$  inside the region  $R$  and all of its coefficients  $w_k$  positive. We assume that  $R$  is a symmetric region (i.e.,  $(x, y) \in R$  implies  $(\pm x, \pm y) \in R$ ) and that  $w(x, y)$  is symmetric in  $x$  and  $y$  and nonnegative. In several recent publications [1]–[5], cubature formulas are computed which have the minimum number of points for their degree. Minimum-point formulas are closely connected with the theory of orthogonal polynomials [6]. This theory, however, is not yet sufficiently developed to give practical results in the case of high degree of exactness ( $d > 7$ ). For  $9 \leq d \leq 15$ , Rabinowitz and Richter [1] have computed perfectly symmetric formulas, by solving a system of nonlinear equations. For  $d = 9$ , the number of points in their formulas is  $N = 20$ .

Until now, no formulas of degree 9 with less than 20 points were known for the square; for the circle, only one 19-point formula is computed by Albrecht [7]. In this note, we describe a numerical method for the construction of 18- and 19-point formulas, if they exist.

It is very likely, but not proved, that for the square and the circular domain with weight function  $w(x) \equiv 1$  the 18-point formulas constructed in this way are minimum-point formulas.

**2. Method of Construction.** Since the region and the weight function are symmetric, it is reasonable to consider only symmetric formulas. Firstly, we construct 19-point formulas. As we shall show further, these formulas must have at least 12 points  $(x_k, y_k)$  with  $x_k \neq 0$  and  $y_k \neq 0$ . We consider then the formula,

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$$\iint_R w(x, y) f(x, y) dx dy$$

$$(2) \cong \sum_{k=1}^3 w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k, -y_k) + f(-x_k, -y_k)]$$

$$+ w_4 [f(x_4, 0) + f(-x_4, 0)] + \sum_{k=5}^6 w_k [f(0, y_k) + f(0, -y_k)] + w_7 f(0, 0),$$

where  $x_k > 0$  and  $y_k > 0$ .

For the computation of the coefficients  $w_k$  and the points  $(x_k, y_k)$  of (2), we have the following set of equations

$$(3.a) \quad 4 \sum_{k=1}^3 w_k x_k^{2\alpha} y_k^{2\beta} = m_{2\alpha, 2\beta}, \quad 0 < \alpha + \beta \leq 4,$$

$$(3.b) \quad 4 \sum_{k=1}^3 w_k x_k^{2\alpha} + 2w_4 x_4^{2\alpha} = m_{2\alpha, 0}, \quad 0 < \alpha \leq 4,$$

$$(3.c) \quad 4 \sum_{k=1}^3 w_k y_k^{2\beta} + 2 \sum_{k=5}^6 w_k y_k^{2\beta} = m_{0, 2\beta}, \quad 0 < \beta \leq 4,$$

$$(3.d) \quad 4 \sum_{k=1}^3 w_k + 2 \sum_{k=4}^6 w_k + w_7 = m_{0, 0}.$$

Here  $\alpha$  and  $\beta$  are natural numbers and

$$m_{\alpha, \beta} = \iint_R w(x, y) x^\alpha y^\beta dx dy.$$

Firstly, we choose  $x_2, x_3$  and  $y_3$  arbitrarily. The system of six equations (3.a) can then be solved explicitly

$$(4) \quad w_3 = \frac{(m_{2,4}^2 m_{6,2} + m_{4,4}^2 m_{2,2} - 2m_{4,4} m_{2,4} m_{4,2} - m_{6,2} m_{2,2} m_{2,6} + m_{4,2}^2 m_{2,6})}{\{[(m_{2,4}^2 - m_{2,2} m_{2,6}) x_3^4 + (m_{4,2}^2 - m_{6,2} m_{2,2}) y_3^4$$

$$+ 2(m_{4,2} m_{2,6} - m_{4,4} m_{2,4}) x_3^2 + 2(m_{2,4} m_{6,2} - m_{4,4} m_{4,2}) y_3^2$$

$$+ 2(m_{4,4} m_{2,2} - m_{2,4} m_{4,2}) x_3^2 y_3^2 + m_{4,4}^2 - m_{6,2} m_{2,6}]\} 4x_3^2 y_3^2},$$

$$x_1^2 = (m_{4,2}^* x_2^2 - m_{6,2}^*) / (m_{2,2}^* x_2^2 - m_{4,2}^*),$$

$$y_1^2 = (m_{2,4}^* x_2^2 - m_{4,4}^*) / (m_{2,2}^* x_2^2 - m_{4,2}^*),$$

$$y_2^2 = [(m_{4,4}^* m_{2,2}^* - m_{2,4}^* m_{4,2}^*) x_2^2 + m_{2,4}^* m_{6,2}^* - m_{4,4}^* m_{4,2}^*] / (m_{6,2}^* m_{2,2}^* - m_{4,2}^*{}^2),$$

$$w_1 = (m_{2,2}^* x_2^2 - m_{4,2}^*)^2 / [4x_1^2 y_1^2 (m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)],$$

$$w_2 = (m_{6,2}^* m_{2,2}^* - m_{4,2}^*{}^2) / [4x_1^2 y_1^2 (m_{2,2}^* x_2^4 - 2m_{4,2}^* x_2 + m_{6,2}^*)],$$

where  $m_{\alpha,\beta}^* = m_{\alpha,\beta} - 4w_3^2 x_3^\alpha y_3^\beta$ . Formula (4) shows that the coefficient  $w_3$  corresponding to the arbitrarily chosen point  $(x_3, y_3)$  is independent of the position of the other points and also different from zero (except perhaps for exceptional regions or weight functions). This means that a symmetric formula of degree 9 requires at least three points in each quadrant of the region.

We consider now the set of four equations (3.b) into which we substitute the free parameters  $x_2, x_3$  and  $y_3$  and the computed values  $w_1, x_1, y_1, w_2$  and  $y_3$ . The first two of these equations can be used for the computation of  $w_4$  and  $x_4$ . The remaining two equations of this set are then considered as a system of two simultaneous equations in the unknowns  $x_2, x_3$  and  $y_3$ . This will generally leave one free parameter, say  $y_3$ . The system of nonlinear equations in the unknowns  $x_2$  and  $x_3$  must be solved numerically.

The parameters  $y_5, y_6, w_5$  and  $w_6$  are then computed by solving the system (3.c). The numbers  $y_5^2$  and  $y_6^2$  are the roots of the quadratic equation,

$$(m_{0,4}^{*2} - m_{0,6}^* m_{0,2}^*)z^2 + (m_{0,2}^* m_{0,8}^* - m_{0,4}^* m_{0,6}^*)z + (m_{0,6}^{*2} - m_{0,4}^* m_{0,8}^*) = 0,$$

while

$$w_5 = (m_{0,2}^* y_6^2 - m_{0,4}^*) / [2y_5^2 (y_6^2 - y_5^2)]$$

and

$$w_6 = (m_{0,4}^* - m_{0,2}^* y_5^2) / [2y_6^2 (y_6^2 - y_5^2)]$$

where

$$m_{0,\beta}^* = m_{0,\beta} - 4 \sum_{k=1}^3 w_k y_k^\beta.$$

Finally,  $w_7$  is computed from (3.d).

From this method of solution, we conclude that there are generally more solutions, since  $y_3$  is still a free parameter. However, it is not impossible that several or even all solutions are complex-valued. In this last case, a 19-point formula of the form (2) does not exist. However, there may still exist a formula of the form,

$$\begin{aligned} & \iint_R w(x, y) f(x, y) dx dy \\ (5) \quad & \cong \sum_{k=1}^4 w_k [f(x_k, y_k) + f(-x_k, y_k) + f(x_k - y_k) + f(-x_k, -y_k)] \\ & + w_5 [f(x_5, 0) + f(-x_5, 0)] + w_6 f(0, 0). \end{aligned}$$

In order to obtain an 18-point formula of degree 9, we consider  $w_7$  as a function of  $y_3$  and we solve the equation  $w_7(y_3) = 0$  (if there exists a real-valued solution). We conjecture that the 18-point formulas computed in this way are minimum-point formulas.

For several regions and weight functions, we have carried out numerical experiments. We summarize the most important results in the following section.

**3. Some Results.** (i) For the square  $R = C_2 = \{(x, y): -1 \leq x, y \leq 1\}$  and  $w(x, y) \equiv 1$  infinitely many 'good' 19-point formulas of degree 9 exist. There are also

at least two 'good' 18-point formulas, the parameters of which we give in Tables 1 and 2 to 20 significant digits. The points of both formulas are common zeros of three orthogonal polynomials of degree 5. These orthogonal polynomials are  $P_0 + \lambda_1 P_2$ ,  $P_0 + \lambda_2 P_4$  and  $P_1 + \mu_1 P_3 + \mu_2 P_5$  where, for the first formula,

$$\begin{aligned}\lambda_1 &= 0.24819696, & \lambda_2 &= 0.25574007, \\ \mu_1 &= 0.00469095, & \mu_2 &= -0.96079906,\end{aligned}$$

and for the second formula,

$$\begin{aligned}\lambda_1 &= -0.22003380, & \lambda_2 &= -0.30398642, \\ \mu_1 &= -0.33809909, & \mu_2 &= -1.33579356,\end{aligned}$$

and where

$$\begin{aligned}P_0(x, y) &= x^5 - 10x^3/9 + 5x/21, \\ P_1(x, y) &= x^4y - 6x^2y/7 + 3y/35, \\ P_2(x, y) &= x^3y^2 - x^3/3 - 3xy^2/5 + x/5, \\ P_3(x, y) &= P_2(y, x), \\ P_4(x, y) &= P_1(y, x), \\ P_5(x, y) &= P_0(y, x),\end{aligned}$$

are the basic orthogonal polynomials for  $C_2$ .

(ii) For the circle  $R = S_2 = \{(x, y): x^2 + y^2 \leq 1\}$  with  $w(x, y) \equiv 1$ , infinitely many 'good' 19-point formulas of degree 9 exist. We have also computed one 18-point formula which has, however, four of its points outside  $S_2$  (see Table 3). The points of this formula are the common zeros of the orthogonal polynomials  $P_0 + \lambda_1 P_2$ ,  $P_0 + \lambda_2 P_4$  and  $P_1 + \mu_1 P_3 + \mu_2 P_5$  with

$$\begin{aligned}\lambda_1 &= 0.56685388, & \lambda_2 &= 0.16924976, \\ \mu_1 &= 0.56433371, & \mu_2 &= -0.65980030,\end{aligned}$$

and where

$$\begin{aligned}P_0(x, y) &= x^5 - x^3 + 3x/16, \\ P_1(x, y) &= x^4y - 3x^2y/5 + 3y/80, \\ P_2(x, y) &= x^3y^2 - x^3/10 - 3xy^2/10 + 3x/80, \\ P_3(x, y) &= P_2(y, x), \\ P_4(x, y) &= P_1(y, x), \\ P_5(x, y) &= P_0(y, x).\end{aligned}$$

(iii) For the entire plane  $R = E_2^2 = \{(x, y): -\infty \leq x, y \leq \infty\}$  with weight function  $w(x, y) = \exp(-x^2 - y^2)$ , infinitely many 'good' 19-point formulas exist. However, we have not found any 18-point formula.

(iv) For the entire plane  $R = E_2^r = \{(x, y): -\infty \leq x, y \leq \infty\}$  with weight function  $w(x, y) = \exp(-(x^2 + y^2)^{1/2})$ , we have not found any real solution of the system of equations (3.a, b, c, d).

A number of the 19-point formulas for  $C_2$ ,  $S_2$  and  $E_2^2$  are tabulated in [8].

TABLE 1. First 18-point formula for the square

k	$x_k$	$y_k$	$w_k$
1	0.87980721399752853896	0.92797961509268528861	0.68416522462309305679 (-1)
2	0.50445910315479838456	0.75347199103161505380	0.27903384209687301395
3	0.91531235408227324183	0.42299357094876513066	0.16806533822999587126
4	0.57882826011929170546	0	0.4092735955433144329
5	0	0.97700090158004246059	0.10648011781560231854
6	0	0.39364057271848893512	0.45321488105170985638

TABLE 2. Second 18-point formula for the square

k	$x_k$	$y_k$	$w_k$
1	0.93742666622066710914	0.94145119299928430974	0.42853317248897088536 (-1)
2	0.57077001686857404415	0.79214654516847247531	0.25788406360659644304
3	0.89774224179848572970	0.40001733897633692860	0.19397744037003970872
4	0.49471787965159623409	0	0.45212398131214854997
5	0	0.98085697194664054422	0.10243215270991495821
6	0	0.48311469619727965642	0.45601422352687001122

TABLE 3. 18-point formula for the circle

k	$x_k$	$y_k$	$w_k$
1	0.86686876801492291622	0.28376671812094800827	0.12937261598422958670
2	0.63925306939199114680	0.95409639862933054563	0.77785540900483355115 (-2)
3	0.48645191470776426796	0.63982457013387676359	0.22713305094453060651
4	0.51286789206607718656	0	0.32090673961381781518
5	0	0.88859953503035797854	0.16042870730308439624
6	0	0.35335517369353007690	0.36089243784037735036

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