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On the Largest Prime Divisor of an Odd Perfect Number. II

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Abstract. It is proved here that every odd perfect number has a prime factor greater than 100110.

If n is an element of the (possibly empty) set of odd perfect numbers, then it is well known that

$$n = p_0^{\alpha_0} \cdot p_1^{\alpha_1} \cdots p_t^{\alpha_t}, \tag{1}$$

where the p_i are distinct primes, $p_0 \equiv \alpha_0 \equiv 1 \pmod 4$, and $2|\alpha_i$ if $i > 0$. In [2], it was proved that at least one of the p_i exceeds 11200. Our purpose here is to improve this bound by proving the following:

THEOREM. *If n is odd and perfect, then n has a prime factor which exceeds* 100110.

The method of proof is similar to that employed in [2], and we shall not give the details here. We shall, however, explain our strategy and exemplify the arguments which are used. The complete proof [1] has been deposited in the UMT file.

The proof is by *reductio ad absurdum.* Thus, we assume that $p_i < 100110$ for every p_i in (1) and show that this assumption is untenable. Since n is perfect $\sigma(n) = 2n$, and since $\sigma(n)$ is multiplicative,

$$2n = \prod_{i=0}^{t} \sigma(p_i^{\alpha_i}) = \prod_{i=0}^{t} \prod_{d} F_d(p_i). \tag{2}$$

Here F_d is the dth cyclotomic polynomial, and d runs over the divisors of $\alpha_i + 1$ which exceed 1. d assumes the value 2 if and only if $i = 0$. We see immediately that the set of p_i in (1) is identical with the set of odd prime divisors of the $F_d(p_i)$ in (2). In particular, recalling our assumption, we note that all the prime factors of each $F_d(p_i)$ must be less than 100110.

For a given odd prime p we shall say that the prime Q is $(p;\ 100110)$-acceptable or simply *p-acceptable* if every prime divisor of $F_Q(p)$ is less than 100110. According to a result of Kanold [3, (21) Satz], if $Q > 50053$, then Q is unacceptable for every odd prime. We shall say that p is *inadmissible* if no Q is p-acceptable. ($Q = 2$ is taken into consideration only if it is possible that $p = p_0$.)

Our proof is in two stages, and we show first that n is not divisible by certain "small" primes.

LEMMA. *If every prime in the factorization of the odd perfect number n is less*

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than 100110, *then n is not divisible by any prime in the set V where* $V = \{3, 5, 7, 11, 13, 19, 23, 31, 37, 61, 127, 131, 151, 1093\}$.

The proof of this lemma goes as follows: Assuming that $p|n$ (which we wish to disprove), we find all p-acceptable primes and then factor $F_Q(p)$; from (2) $F_Q(p)|2n$ for at least one p-acceptable prime Q *and* each odd prime divisor of $F_Q(p)$ divides n; for *each* acceptable $F_Q(p)$, a single prime divisor is selected and *its* acceptable primes are determined; this procedure is iterated and a finite tree is generated (*finite*, since each prime on which we branch is less than 100110 *and* its acceptable primes are bounded by 50053); each path through the tree terminates at a node corresponding to either an inadmissible prime or some other contradiction so that $p \nmid n$. *A priori,* a third type of *terminal* node might be encountered—one corresponding to an admissible prime r such that every odd prime divisor of each r-acceptable cyclotomic number has already been branched upon on the path joining p to r, in which case our procedure fails. We encountered no such nodes, and fortunately most of the trees generated were small. We illustrate by showing that neither 1093 nor 151 divides n, and begin by proving:

(A) If $613|n$, then $613 = p_0$. The only odd 613-acceptable primes are 3 and 5 and $F_3(613) = 3 \cdot 7 \cdot 17923$, $F_5(613) = 131 \cdot 20161 \cdot 53551$. Therefore, if $613|n$ and $613 \neq p_0$ then $17923|n$ or $53551|n$. Since both 17923 and 53551 are inadmissible, our result follows.

(B) $1093 \nmid n$. Only 2 is 1093-acceptable and $F_2(1093) = 2 \cdot 547$. Therefore, if $1093|n$, then $547|n$ also. Only 3 is 547-acceptable and $F_3(547) = 3 \cdot 163 \cdot 613$. Therefore, $1093 = p_0$ *and* (from (A)) $613 = p_0$. We have reached a contradiction.

(C) $151 \nmid n$. For, only 3 is 151-acceptable and $F_3(151) = 3 \cdot 7 \cdot 1093$, so that if $151|n$, then $1093|n$, which contradicts (B).

To describe the second stage of our proof, we need several more definitions. Let q be the smallest prime divisor of n and let $W(q)$ denote the set of primes which are not less than q. For a given prime p, we shall say that the prime Q is $(p; q; 100110)$-feasible or simply (p, q)-*feasible* if Q is p-acceptable and if every odd prime divisor of $F_Q(p)$ belongs to the set $W(q) \cap V'$ where V' denotes the complement of V with respect to the set of all primes. (Of course, for each p_i in (2), each prime divisor of $\alpha_i + 1$ must be (p_i, q)-feasible.) If p cannot be p_0, we omit $Q = 2$ from consideration. If no Q is (p, q)-feasible, we shall say that p is q-*impossible.*

Now, according to the table in [4], $q < 307$ since otherwise n would have a prime factor which exceeds 100549. But (see [1]) except for the elements of the set $T = \{17, 41, 59, 67, 71, 79, 89, 101, 149, 167, 173, 197, 293\}$ every odd prime r less than 307 is either r-impossible or belongs to V, so that $q \in T$. Using basically the method described above for the proof of our lemma, we complete the proof of our theorem by showing that no prime in T is q. We illustrate by proving:

(a) $q \neq 17$. For, only 3 and 5 are (17, 17)-feasible. But $F_3(17) = 307$, only 5 is (307, 17)-feasible, $1051|F_5(307)$, and 1051 is 17-impossible. $F_5(17) = 88741$, only 2 is (88741, 17)-feasible, $44371|F_2(88714)$, and 44371 is 17-impossible.

Concluding Remarks. If P is the largest prime divisor of the odd perfect number n, then a "good" bound on P is very helpful if one is investigating such questions as

"How large is n?" or "How many prime divisors does n have?". This is the motivation for the present paper. It is obvious that by modifying appropriately the definitions of p-acceptable, (p, q)-feasible, etc., and expending the requisite effort and computer time, one could very probably improve our lower bound on P. The present investigation consumed approximately 6.5 hours of CDC 6400 time, most of which was devoted to verifying that, for each prime on which we branched, almost all $Q \leqslant 50053$ were unacceptable. The complete factorizations of all p-acceptable $F_Q(p)$ encountered are given in Table I in [1]. We do not intend to pursue this research further and would hope that if someone else does that he aim for a lower bound on P of at least 10^6.

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