

Numerical Solution for Eigenvalues and Eigenfunctions of a Hermitian Kernel and an Error Estimate

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Abstract. New error estimates for eigenvalues of symmetric integral equations are obtained. These estimates are applicable to a more general class of integration methods and, in many cases, are better than those of Wielandt. For every eigenvalue, a numerical solution for the corresponding eigenfunction is also obtained. Whenever the exact eigenvalue happens to be simple, an error estimate for the corresponding eigenfunction is also derived.

1. Introduction. Let $K(x, t)$ be a Hermitian kernel defined in $I \times I$, where $I \equiv [a, b]$, i.e., $K(t, x) = \overline{K(x, t)}$, such that

$$F(x) \equiv \int_a^b |K(x, t)|^2 dt \text{ is bounded in } I;$$

then all the *characteristic values* μ_i of $K(x, t)$ are real and there exists an orthonormal set $\{y_i(x)\}$ of *characteristic functions* [5], i.e.,

$$(1) \quad \int_a^b K(x, t)y_i(t) dt = \mu_i y_i(x), \quad (y_i, y_j) = \delta_{ij},$$

where $(u, v) \equiv \int_a^b u(x)\overline{v(x)} dx$ is the scalar product of two complex functions $u(x)$, $u(x) \in L_2(I) \equiv \{z(x) | (z, z) < \infty\}$.

Further, let S be a rule of numerical integration with weights $w_{in} > 0$ and nodes $x_{in} \in I$, $i = 1, \dots, n$, by which the approximation $\int_a^b f(x) dx \approx \sum_{i=1}^n w_{in} f(x_{in})$ is made.

To obtain a numerical solution for the characteristic values of $K(x, t)$, Wielandt [9] replaced the original problem by the sequence of eigenproblems

$$(2) \quad K^{(n)} y_i^{(n)} = \mu_{in} y_i^{(n)}, \quad K_{ij}^{(n)} \equiv w_{jn} K(x_{in}, x_{jn}), \quad i, j = 1, \dots, n,$$

with real μ_{in} and n linearly independent eigenvectors $y_i^{(n)}$, for a class of integration rules possessing the properties

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n w_{in} f(x_{in}) = \int_a^b f(x) dx \quad \text{for every } f(x) \in C(I),$$

$$(3) \quad \sum_{i=1}^n w_{in} = b - a;$$

the eigenvalues μ_{kn} , $k = 1, \dots, n$, are then taken by Wielandt as approximations, which also converge as $n \rightarrow \infty$, to the corresponding characteristic values of $K(x, t)$.

To specify this correspondence, the following assumptions are made:

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Let $V \equiv \{\alpha_1, \dots, \alpha_m\}$ be a subset of the set R of all eigenvalues of a square matrix A or of all characteristic values of a kernel $F(x, t)$ defined in $I \times I$, and let $W \equiv \{z^2 \mid z \in V\}$; then

(a) if $\alpha_1, \dots, \alpha_m$ are the m largest (smallest) real elements of R such that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ ($\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m$), then every $\alpha_i \neq \alpha_m$ with multiplicity $r_i > 1$ occurs r_i times in V ,

(b) if $\alpha_1, \dots, \alpha_m$ are the m real elements of R the largest modulus such that $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_m|$ and there are r_i real elements of R of modulus $|\alpha_i|$, then every $\alpha_i^2 \neq \alpha_m^2$ occurs r_i times in W .

The problem which arises now is what is the best error estimate for the eigenvalues μ_{kn} of (2). In this context, and with the above assumptions, Wielandt obtained for those integration rules, which we shall call *convergent with respect to $K(x, t)$* —i.e. the sequence

$$(4) \quad \eta_n(x, t) \equiv \sum_{i=1}^n w_{in} K(x, x_{in}) K(x_{in}, t) - \int_a^b K(x, z) K(z, t) dz$$

of the error functions for $f(z) \equiv K(x, z) K(z, t)$ converges to 0 uniformly in $I \times I$, the following result:

Let $\mu_{1n}^+ \geq \mu_{2n}^+ \geq \dots \geq \mu_{rn}^+ > 0 > \mu_{sn}^- \geq \dots \geq \mu_{2n}^- \geq \mu_{1n}^-$ be the r largest positive and the s smallest negative eigenvalues of (2), and let

$$\mu_1^+ \geq \mu_2^+ \geq \dots \geq \mu_r^+ > 0 > \mu_s^- \geq \dots \geq \mu_2^- \geq \mu_1^-$$

be the corresponding characteristic values of $K(x, t)$; then

$$\mu_i^+ = \lim_{n \rightarrow \infty} \mu_{in}^+, \quad \mu_j^- = \lim_{n \rightarrow \infty} \mu_{jn}^-, \quad i = 1, \dots, r, \quad j = 1, \dots, s,$$

and this convergence is uniform in i and j , i.e.,

$$\sigma_{kn} \equiv |\mu_{kn} - \mu_k| \leq q_n, \quad \lim_{n \rightarrow \infty} q_n = 0,$$

where either

$$\mu_{kn} = \mu_{kn}^+, \quad \mu_k = \mu_k^+, \quad \text{or} \quad \mu_{kn} = \mu_{kn}^-, \quad \mu_k = \mu_k^-.$$

Baker [2] obtained convergence properties of a similar type for simple characteristic values of $K(x, t)$. The best estimate obtained by Wielandt is $q_n = O(\sqrt{\epsilon_n})$, where $\epsilon_n \equiv \max_{I \times I} |\eta_n(x, t)|$ and $\eta_n(x, t)$ is defined by (4), whereas that of Baker is $q_n = O(\max w_{in})$. Other authors ([1], [3]) obtained better bounds, but only for the distance of every eigenvalue μ_{kn} to the nearest characteristic value of $K(x, t)$. In this paper improved estimates of the form (see Theorem 1 at the beginning of Section 4)

$$\sigma_{kn} = [\max(|\mu_{kn}|, |\mu_k|)]^{-1} \rho_n, \quad \rho_n = O(\epsilon_n),$$

are obtained, which generalizes Wielandt's convergence theorems for all integration rules which are convergent with respect to $K(x, t)$ and satisfy (3). Moreover, the new result enables application of integration rules, which are convergent with respect to $K(x, t)$, to kernels which exhibit singular behavior in $I \times I$ and for which, therefore, no solution can be found within the scope of Wielandt's and Baker's papers (see Example 2 in Sec-

tion 2). As a consequence, an error estimate for the numerical solution of (1), convergent to 0 uniformly in I for every integration rule which is convergent with respect to $K(x, t)$, is derived. The new error estimates for eigenvalues are to be interpreted as follows: our estimates are better than those of Wielandt for the first m_n eigenvalues μ_{kn} such that $\max(|\mu_{kn}|, |\mu_k|) > C\sqrt{\epsilon_n}$ for some $C > 0$, where the sequence m_n tends to infinity; for other eigenvalues both our and Wielandt's estimates are of the same order of magnitude, namely $O(\sqrt{\epsilon_n})$, and ours are not necessarily better.

2. Numerical Results. To illustrate the superiority of the new error estimates given by Theorem 1 in Section 4, two numerical examples are presented. To the second of our examples Wielandt's method does not apply.

In the tables of results given below, μ_{in}^+ and μ_{in}^- are the eigenvalues defined in Theorem 2 near the end of Section 4, whereas $y_{in}^+(x)$ and $y_{in}^-(x)$ are the numerical solutions for characteristic functions corresponding to μ_{in}^+ and μ_{in}^- , respectively, obtained by the procedure described at the end of Section 4. The improved error estimates are those described in Section 5. The error estimates for $y_{in}^+(x)$ and $y_{in}^-(x)$ are those obtained by application of the remark concluding the discussion of Theorem 3 in Section 4.

Example 1. The integral equation is

$$\int_0^1 \max(x, t)y(t) dt = \mu y(x).$$

Characteristic values and characteristic functions are, respectively:

$$R^2 \text{ and } \frac{\sqrt{2} \cosh Rx}{\cosh R}, \text{ where } R \text{ is the positive root of the equation } z \tanh z = 1;$$

$$-r_N^2 \text{ and } \frac{\sqrt{2} \cos r_N x}{\cos r_N}, \quad N = 1, 2, \dots, \text{ where } 0 < r_1 < r_2 < \dots \text{ are the positive roots of the equation } z \tan z + 1 = 0.$$

The integration rule S mentioned at the introduction is the trapezoidal rule.

To obtain α_n, β_n and γ_n , as defined in (24), put

$$A_n(z) \equiv (n - 1)z - [(n - 1)z], \quad B_n(z) \equiv 1 - A_n(z), \quad C_n(z) \equiv A_n(z)B_n(z),$$

$$D_n(z) \equiv A_n(z) - B_n(z), \quad h \equiv (n - 1)^{-1}, \quad F_n(z) \equiv 1 - z - C_n(z)[3z - hD_n(z)];$$

then

$$\eta_n(x, t) = \frac{h^2}{6} \begin{cases} 3tC_n(x) + F_n(t), & x \leq t, \\ 3xC_n(t) + F_n(x), & x \geq t, \end{cases}$$

which after a simple, but lengthy, calculation yields

$$\alpha_n^2 = \frac{h^4}{18} \sum_{k=1}^{n-1} \int_{(k-1)h}^{kh} t \{tG_n(t) + hC_n(t)D_n(t)[G_n(t) + 0.3tC_n(t)] + F_n^2(t)\} dt,$$

where $G_n(t) \equiv 0.3t + F_n(t)$, and

$$\int_0^1 \eta_n^2(x, x_{in}) dx = \int_0^{x_{in}} \eta_n^2(x, x_{in}) dx + \int_{x_{in}}^1 \eta_n^2(x, x_{in}) dx.$$

Each of the above summands is evaluated by the closed Newton-Cotes formula with 7 points. The remark at the end of Section 4 is applied with $p = L = 1$.

For comparison with Wielandt's results the error estimates for the negative eigenvalues μ_{in}^- , together with error estimates for the numerical solutions for characteristic functions, are presented in the following table:

TABLE 1

Case	l	Error estimate for μ_{in}^- by Theorem 1	Improved error estimate for μ_{in}^-	Error estimate for $y_{in}^-(x)$	Actual error for μ_{in}^-	Actual maximal error for $y_{in}^-(k/N)$, $k = 0, 1, \dots, N$	Wielandt's estimate
$n = 101$ $N = 200$	1	$7.34 \cdot 10^{-5}$	$7.34 \cdot 10^{-5}$	0.0067	$1.26 \cdot 10^{-5}$	0.00011	0.00539
	2	$6.45 \cdot 10^{-4}$	$3.51 \cdot 10^{-4}$	0.872	$9.22 \cdot 10^{-6}$	0.000466	
	3	0.00153	$8.15 \cdot 10^{-4}$		$8.72 \cdot 10^{-6}$	0.00104	
$n = 201$ $N = 1000$	1	$1.836 \cdot 10^{-5}$	$1.836 \cdot 10^{-5}$	0.00167	$3.14 \cdot 10^{-6}$	$2.77 \cdot 10^{-5}$	0.00269
	2	$1.61 \cdot 10^{-4}$	$8.78 \cdot 10^{-5}$	0.2073	$2.306 \cdot 10^{-6}$	$1.17 \cdot 10^{-4}$	
	3	$3.75 \cdot 10^{-4}$	$2.04 \cdot 10^{-4}$		$2.18 \cdot 10^{-6}$		
	4	$6.85 \cdot 10^{-4}$	$3.66 \cdot 10^{-4}$		$1.628 \cdot 10^{-5}$		

It is to be noted that as the initial error estimates for the eigenvalues μ_{in}^- tend to grow with l , they are better than those of Wielandt only for some first eigenvalues. To obtain a comparable error estimate unobtainable by Theorem 1 for other eigenvalues, the bound q_n defined by (26) with the optimal $C = \sqrt{0.5(1 + \sqrt{5})}$ can be taken.

Example 2. The integral equation is

$$\int_0^1 (1 + i\sqrt{x} - i\sqrt{t})^{-1} y(t) dt = \mu y(x).$$

The exact solution is unknown.

The integration rule, which is derived by the transformation $u = z^2$ for the integral $\int_0^1 K(x, z)K(z, t) dz$ and application of the Gauss quadrature with weights ω_{in} and nodes ξ_{in} , $i = 1, \dots, n$, is defined by

$$w_{in} \equiv 2\omega_{in}\xi_{in}, \quad x_{in} \equiv \xi_{in}^2, \quad i = 1, \dots, n;$$

therefore, using our definition (4) [7, p. 48],

$$\begin{aligned} \eta_n(x, t) &= 2c_n \left[\frac{\partial^{2n}}{\partial u^{2n}} uK(x, u^2)K(u^2, t) \right]_{u=\xi} \\ &= \frac{2(2n)!c_n}{\sqrt{x} - \sqrt{t} - 2i} \left[\frac{\sqrt{x} - i}{(\xi - \sqrt{x} + i)^{2n+1}} - \frac{\sqrt{t} + i}{(\xi - \sqrt{t} - i)^{2n+1}} \right], \end{aligned}$$

where $c_n = [(2n)^2(2n + 1)!]^{-1}$ and $0 < \xi = \xi(x, t) < 1$, and consequently

$$|\eta_n(x, t)| \leq \binom{2n}{n}^{-2} (2n + 1)^{-1} (\sqrt{1 + x} + \sqrt{1 + t}).$$

The error estimates, with those for $y_{in}^+(x)$ obtained by application of the remark at the end of Section 4 with $p = \frac{1}{2}$ and $L = 1$, are given in the table below:

TABLE 2

n	l	Error estimate for μ_{ln}^+ by Theorem 1	Improved error estimate for μ_{ln}^+	Error estimate for $y_{ln}^+(x)$
6	1	$2.31 \cdot 10^{-7}$	$2.31 \cdot 10^{-7}$	$8.8 \cdot 10^{-7}$
	2	$1.015 \cdot 10^{-5}$	$5.08 \cdot 10^{-6}$	0.00383
	3	$2.12 \cdot 10^{-4}$	$1.033 \cdot 10^{-4}$	
8	1	$9.1 \cdot 10^{-10}$	$9.1 \cdot 10^{-10}$	$3.41 \cdot 10^{-9}$
	2	$4 \cdot 10^{-8}$	$2 \cdot 10^{-8}$	$1.47 \cdot 10^{-5}$
	3	$8.15 \cdot 10^{-7}$	$4.08 \cdot 10^{-7}$	0.11

The approximations for $n = 9$ rounded to 10 digits are:

$$\mu_{1n}^+ \approx 0.9543482459, \quad \mu_{2n}^+ \approx 0.0434068611, \quad \mu_{3n}^+ \approx 0.0021321407.$$

3. Numerical Solution for a Characteristic Function. Since the new results, presented in Section 2, refer also to an error estimate for the corresponding characteristic function, an appropriate definition of the approximate solution for a characteristic function, which converges to the corresponding characteristic function, is to be given. To obtain such a definition, observe first that by the similarity relation between the matrix $K^{(n)}$ of (2) and the Hermitian matrix H with $H_{ij} \equiv K(x_{in}, x_{jn})\sqrt{w_{in}w_{jn}}$, $i, j = 1, \dots, n$, the eigenvector $y_k^{(n)}$ is related to the corresponding eigenvector z_k of H by $z_{ki} = y_{ki}^{(n)}\sqrt{w_{in}}$, $i = 1, \dots, n$. Further, define a new scalar product $(u, v)_n$ of two vectors u, v in C_n —the n -dimensional complex Euclidean space—and a new norm $|u|_n$ in C_n , by

$$(5) \quad (u, v)_n \equiv \sum_{i=1}^n w_{in} u_i \bar{v}_i, \quad |u|_n \equiv \sqrt{(u, u)_n},$$

and denote by $\|f\| \equiv \sqrt{(f, f)}$ the norm of a complex function $f(x)$; therefore, if the eigenvectors z_k , $k = 1, \dots, n$, of H are chosen so as to form an orthonormal set, then

$$(6) \quad (y_p^{(n)}, y_q^{(n)})_n = \delta_{pq}, \quad p, q = 1, \dots, n.$$

For every eigenvector $y_k^{(n)}$ of (2) with $\mu_{kn} \neq 0$, define now the *numerical solution* $y_{kn}(x)$ for a characteristic function generated by $y_k^{(n)}$, which also satisfies $y_{kn}(x_{in}) = y_{ki}^{(n)}$, $i = 1, \dots, n$, as

$$(7) \quad y_{kn}(x) \equiv \mu_{kn}^{-1} \sum_{j=1}^n w_{jn} y_{kj}^{(n)} K(x, x_{jn}).$$

It is natural to expect the difference between the two sides of (1), with μ_k and $y_k(x)$ replaced by μ_{kn} and $y_{kn}(x)$, respectively, to be expressible in terms of the error function (4). In fact,

$$(8) \quad \mu_{kn} y_{kn}(x) - \int_a^b K(x, t) y_{kn}(t) dt = \mu_{kn}^{-1} \sum_{j=1}^n w_{jn} y_{kj}^{(n)} \eta_n(x, x_{jn}),$$

where $\eta_n(x, t)$ is defined by (4).

Let, further, $\{y_m^*(x)\}$, $m = 1, \dots, r$, form an orthonormal base of all characteristic functions of $K(x, t)$ corresponding to μ_k ; then for every n with $\mu_{kn} \neq 0$, there exist coefficients $c_{km}^{(n)}$, $m = 1, \dots, r$, such that the error function

$$e_{kn}(x) \equiv y_{kn}(x) - \sum_{m=1}^r c_{km}^{(n)} y_m^*(x)$$

is of minimal norm. In fact,

$$c_{km}^{(n)} = (y_{kn}, y_m^*), \quad m = 1, \dots, r;$$

and, consequently,

(9) $(e_{kn}, y) = 0$ for every characteristic function $y(x)$ of $K(x, t)$ corresponding to μ_k .

The functions $e_{kn}(x)$ and $\tilde{y}_{kn}(x) \equiv y_{kn}(x) - e_{kn}(x)$ are called *the error function and the characteristic function*, respectively, associated with $y_{kn}(x)$. Now, if the approximate numerical solution $y_{kn}^*(x)$ for a characteristic function is taken to be of norm 1, i.e., $y_{kn}^*(x) = \|y_{kn}\|^{-1} y_{kn}(x)$, it can be shown that the characteristic function $Y_{kn}(x)$ of norm 1 corresponding to μ_k such that the error function $e_{kn}^*(x) \equiv y_{kn}^*(x) - Y_{kn}(x)$ is of minimal norm, assumes the form

$$Y_{kn}(x) = \begin{cases} R_{kn}^{-1} \tilde{y}_{kn}(x), & R_{kn} \neq 0, \\ y_i(x) \text{ with } \mu_i = \mu_k, & R_{kn} = 0, \end{cases}$$

where $R_{kn} = \|\tilde{y}_{kn}\|$. Also, since by (9) $\|y_{kn}\|^2 = R_{kn}^2 + \|e_{kn}\|^2$, we have

$$\begin{aligned} e_{kn}^*(x) &= \|y_{kn}\|^{-1} [e_{kn}(x) - (\|y_{kn}\| - R_{kn}) Y_{kn}(x)] \\ (10) \quad &= \|y_{kn}\|^{-1} \left[e_{kn}(x) - \frac{\|e_{kn}\|^2}{\|y_{kn}\| + R_{kn}} Y_{kn}(x) \right]. \end{aligned}$$

4. Error Estimate and Convergence. For the sake of conciseness of presentation, the following definitions are introduced:

μ_k and μ_{kn} , $k = 1, \dots, r$, are the r largest (smallest) characteristic values of $K(x, t)$ and the r largest (smallest) eigenvalues of (2), respectively, such

(11) that $\mu_i \geq \mu_{i+1}$ ($\mu_i \leq \mu_{i+1}$) and $\mu_{in} \geq \mu_{i+1,n}$ ($\mu_{in} \leq \mu_{i+1,n}$), $i = 1, \dots, r - 1$.

In the following, $F(x, t)$ is a kernel defined in $I \times I$.

$U(F)$ and $U_n(F)$ are the set of all characteristic values of $F(x, t)$ and the set

(12) of all eigenvalues of the matrix $F^{(n)}$ with $F_{ij}^{(n)} \equiv w_{jn} F(x_{in}, x_{jn})$, $i, j = 1, \dots, n$, respectively.

(13) $\lambda_k(F)$ and $\lambda_{kn}(F)$ are the k th real elements of $U(F)$ and $U_n(F)$, respectively, in the ordering determined by that of the μ_k and the μ_{kn} in (11).

$M_k(F)$ and $M_{kn}(F)$, $k = 1, \dots, r$, are the moduli of the r elements of $U(F)$

(14) and $U_n(F)$, respectively, of largest modulus, such that $M_i(F) \geq M_{i+1}(F)$ and $M_{in}(F) \geq M_{i+1,n}(F)$, $i = 1, \dots, r - 1$.

$$(15) \quad Q(F, u) \equiv \int_a^b \int_a^b F(x, t) u(t) \overline{u(x)} dx dt,$$

$$(16) \quad Q_n(F, u) \equiv \sum_{i,j=1}^n w_{in} w_{jn} F(x_{in}, x_{jn}) u_j \overline{u_i},$$

$$(17) \quad V_k(F) \equiv \left\{ Y \left| \int_a^b F(x, t) Y(t) dt = \lambda_k(F) Y(x), \|Y\| = 1 \right. \right\},$$

$$(18) \quad V_{kn}(F) \equiv \{z | F^{(n)}z = \lambda_{kn}(F)z, |z|_n = 1\}, \quad \text{where}$$

$$F_{ij}^{(n)} \equiv w_{jn} F(x_{in}, x_{jn}), \quad i, j = 1, \dots, n, \quad \text{and } |z|_n \text{ is defined by (5).}$$

$$(19) \quad \delta_n(F, x, t) \equiv \sum_{i=1}^n w_{in} F(x, x_{in}) F(x_{in}, t) - \int_a^b F(x, z) F(z, t) dz,$$

$$(20) \quad D_n(F, u) \equiv \int_a^b \int_a^b \delta_n(F, x, t) \overline{u(x)} u(t) dx dt,$$

$$(21) \quad D_n^*(F, u) \equiv \sum_{i,j=1}^n w_{in} w_{jn} \delta(F, x_{in}, x_{jn}) u_i \overline{u_j},$$

$$(22) \quad A_{kn}(F) \equiv \left[\max \left\{ \sum_{i=1}^n w_{in} \left| \int_a^b \delta_n(F, x, x_{in}) \overline{u(x)} dx \right|^2 \mid u \in V_k(F) \right\} \right]^{1/2},$$

$$(23) \quad B_{kn}(F) \equiv \left[\max \left\{ \int_a^b \left| \sum_{i=1}^n w_{in} \overline{u_i} \delta_n(F, x_{in}, x) \right|^2 dx \mid u \in V_{kn}(F) \right\} \right]^{1/2};$$

$$\alpha_n \equiv \left[\int_a^b \int_a^b |\eta_n(x, t)|^2 dx dt \right]^{1/2}, \quad \beta_n \equiv \left[\sum_{i,j=1}^n w_{in} w_{jn} |\eta_n(x_{in}, x_{jn})|^2 \right]^{1/2},$$

$$(24) \quad \gamma_n \equiv \left[\sum_{i=1}^n w_{in} \int_a^b |\eta_n(x, x_{in})|^2 dx \right]^{1/2}, \quad \rho_n \equiv \max(\alpha_n, \beta_n),$$

where $\eta_n(x, t)$ is defined by (4).

$$(25) \quad y_{kn}(x) \text{ and } e_{kn}(x) \text{ are, respectively, the function (7) and the error function associated with it as defined in Section 3.}$$

The new error estimates for the eigenvalues obtained in this paper are now summarized in the following two theorems:

THEOREM 1. *If, with Definitions (11) and (24), $\nu_{kn} \equiv \max(|\mu_{kn}|, |\mu_k|) \geq C\sqrt{\rho_n}$ for some $C > 1$, then*

$$(a) \quad |\mu_{kn} - \mu_k| \leq \nu_{kn}^{-1} (\gamma_n + \rho_n) [1 - \nu_{kn}^{-2} \rho_n]^{-1/2} \leq \nu_{kn}^{-1} (\gamma_n + \rho_n) [1 - C^{-2}]^{-1/2},$$

$$(b) \quad |\mu_{1n} - \mu_1| \leq \gamma_n [\nu_{1n}^2 - \rho_n]^{-1/2}.$$

THEOREM 2. *Let $\mu_{1n}^+ \geq \mu_{2n}^+ \geq \dots \geq \mu_{rn}^+ > 0$, $\mu_{1n}^- \leq \mu_{2n}^- \leq \dots \leq \mu_{sn}^- < 0$, be the r largest positive and the s smallest negative eigenvalues of (2), and let $\mu_1^+ \geq \mu_2^+ \geq \dots \geq \mu_r^+ > 0 > \mu_s^- \geq \dots \geq \mu_2^- \geq \mu_1^-$ be the corresponding characteristic values of $K(x, t)$.*

If the integration rule S is convergent with respect to $K(x, t)$ and satisfies (3), then

$$\lim_{n \rightarrow \infty} \mu_{in}^+ = \mu_i^+, \quad \lim_{n \rightarrow \infty} \mu_{jn}^- = \mu_j^-, \quad i = 1, \dots, r, \quad j = 1, \dots, s;$$

and the convergence is uniform in i and j , so that for every $C > 1$,

$$|\mu_{in}^+ - \mu_i^+| \leq q_n, \quad |\mu_{jn}^- - \mu_j^-| \leq q_n, \quad i = 1, \dots, r, \quad j = 1, \dots, s,$$

$$(26) \quad q_n \equiv \max\{C, [C^2 - 1]^{-1/2}\} \sqrt{\gamma_n + \rho_n}, \quad \text{where } \gamma_n \text{ and } \rho_n \text{ are defined by (24).}$$

Theorem 2 is a generalization of Wielandt's results.

The error estimate for the approximate numerical solution of (1) is given by:

THEOREM 3. *The error function defined by (25) satisfies (see Definitions (4), (11) and (12))*

$$|e_{kn}(x)| \leq |\mu_k^{-1} \mu_{kn}^{-1}| \left\{ \sum_{j=1}^n w_{jn} |y_{kj}^{(n)}| \left[\max_I |\eta_n(x, x_{jn})| + q_{kn} I_{jn} \sqrt{F(x)} \right] + |\mu_{kn} - \mu_k| G_n(x) \right\},$$

where

$$F(x) \equiv \int_a^b |K(x, t)|^2 dt, \quad G_n(x) \equiv [F(x) + \eta_n(x, x)]^{1/2},$$

$$(27) \quad q_{kn} \equiv \sup \{ |\mu_{kn} - \lambda|^{-1} |\lambda \in U(K), \lambda \neq \mu_k \},$$

$$I_{jn} \equiv \left[\int_a^b |\eta_n(x, x_{jn})|^2 dx \right]^{1/2}.$$

This bound for $e_{kn}(x)$, and consequently that for the function $e_{kn}^*(x)$ defined by (10), are improvements, by a factor of $O(n^{-1/2})$, of a similar error estimate obtained in [4].

Error estimates for the eigenvalues in special cases are given in Section 5.

An immediate consequence of Theorem 2, analogous to the one which follows from the convergence theorem in [1], is:

If the integration rule S is convergent with respect to $K(x, t)$, then $e_{kn}(x)$ and $e_{kn}^*(x)$ converge to 0 uniformly in I .

Remark. If $K(x, t)$ satisfies a Lipschitz condition of the form $|K(u, t) - K(v, t)| \leq L|u - v|^p$, $0 < p \leq 1$, in $I \times I$, then (see Definition (4))

$$|e_{kn}(x)| \leq |\mu_k^{-1}| \left\{ |\mu_{kn}^{-1}| \sum_{j=1}^n w_{jn} |y_{kj}^{(n)}| \left[\max_I |\eta_n(x, x_{jn})| + q_{kn} I_{jn} \sqrt{F(x)} \right] + |\mu_{kn} - \mu_k| \left[\max_m |y_{km}^{(n)}| + \frac{2^{-p} L}{|\mu_{kn}|} \max_{0 \leq m < n} (x_{m+1,n} - x_{mn})^p \sum_{j=1}^n w_{jn} |y_{kj}^{(n)}| \right] \right\}$$

where $x_{0n} = a$ and $x_{n+1,n} = b$.

Since the estimate for $e_{kn}(x)$ involves an estimate for q_{kn} , it can be found only if the multiplicity of μ_k is known. Such an estimate is obtainable, for instance, when μ_k is a simple characteristic value and in this case we deduce by Theorem 2, Eq. (38) of [6] and Lemma 1 in Section 6:

COROLLARY 1. If μ_k is a simple characteristic value of $K(x, t)$ and the integration rule is convergent with respect to $K(x, t)$ and satisfies (3), then for some choice of eigenvectors $y_k^{(n)}$ such that $|y_k^{(n)}|_n = 1$ (see Definitions (5) and (7)),

$$y_{kn}(x) \rightarrow y_k(x) \quad \text{uniformly in } I, \text{ and so does also } \|y_{kn}\|^{-1} y_{kn}(x).$$

By (7) it follows that

$$\|y_{kn}\|^2 = \mu_{kn}^{-2} \sum_{i,j=1}^n w_{in} w_{jn} \overline{y_{ki}^{(n)}} y_{kj}^{(n)} G(x_{in}, x_{jn})$$

where

$$G(x, t) \equiv \int_a^b \overline{K(z, x)} K(z, t) dz = \int_a^b K(x, z) K(z, t) dz.$$

In the case where $G(x, t)$ cannot be determined exactly, an approximation c_{kn} of $\|y_{kn}\|$ is found by applying some quadrature formula for determining $G(x, t)$ at the points (x_{in}, x_{jn}) ; the approximate solution for a characteristic function is then taken to be $c_{kn}^{-1} y_{kn}(x)$, and the error estimate is

$$\begin{aligned} \tilde{e}_{kn}(x) &= |c_{kn}^{-1} y_{kn}(x) - R_{kn}^{-1} \tilde{y}_{kn}(x)| \leq |(c_{kn}^{-1} - \|y_{kn}\|^{-1}) y_{kn}(x)| + |e_{kn}^*(x)| \\ &= (c_{kn} \|y_{kn}\|)^{-1} |(c_{kn} - \|y_{kn}\|) y_{kn}(x)| + |e_{kn}^*(x)|, \end{aligned}$$

where $e_{kn}^*(x)$ is given by (10).

5. Improved Error Estimates for Simple Characteristic Values and for Positive-Definite Kernels. An error estimate for a simple characteristic value can be improved if the approximate eigenvalue μ_{kn} satisfies the inequality (see Definition (11))

$$|\mu_{kn} - \mu_k| < \min_{i \neq k} |\mu_{kn} - \mu_i|.$$

If the integration rule S is convergent with respect to $K(x, t)$ and satisfies (3), and μ_k is a simple characteristic value, then by Theorem 2 there exists an integer N such that the above inequality holds for $n > N$, and by Lemma 5 (stated in Section 6)

$$|\mu_{kn} - \mu_k| = \inf\{|\mu_{kn} - \lambda| \mid \lambda \in U(K)\} \leq |\mu_{kn}^{-1}| \|y_{kn}\|^{-1} \gamma_n.$$

An error estimate for a positive-definite kernel is obtained from the following theorem:

THEOREM 4. Let $\tilde{\mu}_{kn} \in U_n(K)$ and $\tilde{\mu}_k \in U(K)$, $k = 1, \dots, n$, such that $|\tilde{\mu}_{kn}| = M_{kn}(K)$ and $|\tilde{\mu}_k| = M_k(K)$ (see Definitions (12), (14) and (4)). Then

$$|\tilde{\mu}_{kn}^2 - \tilde{\mu}_k^2| \leq M_1(\eta_n) + M_{1n}(\eta_n), \quad k = 1, 2, \dots,$$

where $\tilde{\mu}_{kn} = 0$ for $k > n$.

COROLLARY 2. If $K(x, t)$ is positive-definite and (see Definitions (11) and (12)), $\mu_{kn} > -\min\{|\lambda| \mid \lambda \in U_n(K)\}$, then, $|\mu_{kn}^2 - \mu_k^2| \leq M_1(\eta_n) + M_{1n}(\eta_n)$.

We also obtain

$$\text{COROLLARY 3. } |\mu_{n+k}| \leq \sqrt{M_1(\eta_n)}, \quad k = 1, 2, \dots$$

6. Discussion of the Theorems. To obtain the final results presented in Theorems 1–4, the following lemmas are necessary using the definitions introduced at the beginning of Section 4:

LEMMA 1. Let $u_n = (u_{n1}, \dots, u_{nn})$ and $A_n = (a_{ij}^{(n)})$ be, respectively, sequences of vectors and $n \times n$ matrices with complex elements. Then (see Definition (5))

(a) $|u_n|_n^2 = \sum_{k=1}^n |(u_n, z_k)_n|^2 = \sum_{k=1}^n |(z_k, u_n)_n|^2$ for every sequence $z_k, k = 1, \dots, n$, satisfying

$$(28) \quad (z_p, z_q)_n = \delta_{pq}, \quad p, q = 1, \dots, n.$$

(b) If the integration rule S satisfies (3), then

$$\lim_{n \rightarrow \infty} \max_i |u_{ni}| = 0 \text{ implies } \lim_{n \rightarrow \infty} \sum_{i=1}^n w_{in} u_{ni} = \lim_{n \rightarrow \infty} \sum_{i=1}^n w_{in} |y_{ki}^{(n)}| u_{ni} = 0, \quad k = 1, 2, \dots,$$

and

$$\lim_{n \rightarrow \infty} \max_{i,j} |a_{ij}^{(n)}| = 0 \text{ implies } \lim_{n \rightarrow \infty} \sum_{i,j=1}^n w_{in} w_{jn} a_{ij}^{(n)} = 0.$$

LEMMA 2. If $\lambda_k \equiv \lambda_k(F) = \lambda_1(F)$, where $F(x, t)$ is a Hermitian kernel defined in $I \times I$, then (see Definitions (13), (22), (17) and (20))

$$\lambda_k(\lambda_k - \lambda_{kn}(F)) \leq A_{kn}(F) \left[1 - \lambda_k^{-2} \max_{V_k(F)} |D_n(F, u)| \right]^{-1/2}.$$

LEMMA 3. If $\lambda_{kn} \equiv \lambda_{kn}(F) = \lambda_{1n}(F)$, where $F(x, t)$ is a Hermitian kernel defined in $I \times I$, then (see Definitions (13), (23), (18) and (21))

$$\lambda_{kn}(\lambda_{kn} - \lambda_k(F)) \leq B_{kn}(F) \left[1 - \lambda_{kn}^{-2} \max_{V_{kn}(F)} |D_n^*(F, u)| \right]^{-1/2}.$$

This lemma is a consequence of Weyl's theorem [8, p. 445]:

LEMMA 4. Let $D(x, t) \equiv F(x, t) - G(x, t)$, where $F(x, t)$ and $G(x, t)$ are Hermitian kernels defined in $I \times I$; then (see Definitions (13), (15) and (16))

- (a) if $Q(D, u) \geq 0$ for every $u(x)$, then $\lambda_k(F) \geq \lambda_k(G), k = 1, 2, \dots;$
- (b) if $Q_n(D, u) \geq 0$ for every $u \in C_n$, then $\lambda_{kn}(F) \geq \lambda_{kn}(G), k = 1, \dots, n.$

The next and last lemma is used to obtain the improved error estimates for simple characteristic values mentioned in Section 5.

LEMMA 5. With Definitions (11), (12), (7) and (24),

$$D_{kn} \equiv \inf\{|\mu_{kn} - \lambda| \mid \lambda \in U(K)\} \leq |\mu_{kn}^{-1}| \|y_{kn}\|^{-1} \gamma_n.$$

This is a slight improvement of the result obtained in [3].

The proofs of Lemmas 1, 4 and 5 are straightforward ([6, Lemmas 1, 5 and 2, respectively]), whereas those of Lemmas 2 and 3 require some special devices ([6, Lemmas 3 and 4, respectively]).

The first four lemmas are used to establish part (a) of Theorem 1 by the following steps:

1. Application of Lemma 2 and part (b) of Lemma 4 to obtain (see Definitions (11), (22), (17) and (20))

$$(29) \quad \mu_k(\mu_k - \mu_{kn}) \leq A_{kn}(L) \left[1 - \mu_k^{-2} \max_{V_k(L)} |D_n(L, u)| \right]^{-1/2},$$

where

$$(30) \quad L(x, t) \equiv K(x, t) - \sum_{p=1}^{k-1} (\mu_p - \mu_k) y_p(x) \overline{y_p(t)}.$$

2. Application of Lemma 3 and part (a) of Lemma 4 to obtain (see Definitions (11), (23), (18) and (21))

$$(31) \quad \mu_{kn}(\mu_{kn} - \mu_k) \leq B_{kn}(L_n) \left[1 - \mu_{kn}^{-2} \max_{V_{kn}(L_n)} |D_n^*(L_n, u)| \right]^{-1/2},$$

where

$$(32) \quad L_n(x, t) \equiv K(x, t) - \sum_{p=1}^{k-1} (\mu_{pn} - \mu_{kn}) y_{pn}(x) \overline{y_{pn}(t)}.$$

3. Bounding of $A_{kn}(L)$, $\max\{|D_n(L, u)| \mid u \in V_k(L)\}$, $B_{kn}(L_n)$ and $\max\{|D_n^*(L_n, u)| \mid u \in V_{kn}(L_n)\}$ in terms of γ_n and ρ_n defined by (24), which is a matter of pure manipulations.

Theorem 2 follows from Lemma 1 and Theorem 1.

Theorem 3 is a consequence of (9) and the Parseval equality (equation of closedness [5, p. 10]) for the function $e_{kn}(x)$.

For the full proof of the above theorems the reader is referred to [6].

Finally, we come to the proof of Theorem 4, which terminates our discussion.

Proof of Theorem 4. The degenerate kernel

$$G_n(x, t) \equiv \sum_{i=1}^n w_{in} K(x, x_{in}) K(x_{in}, t) = \sum_{i=1}^n w_{in} \overline{K(x_{in}, x)} K(x_{in}, t),$$

is Hermitian and $G_n(x, t) = G(x, t) + \eta_n(x, t)$, where

$$G(x, t) \equiv \int_a^b K(x, z) K(z, t) dz,$$

therefore the characteristic values ν_{kn} of $G_n(x, t)$, where $\nu_{1n} \geq \nu_{2n} \geq \dots \geq \nu_{nn} \geq \nu_{n+1,n} = \dots = 0$, are related to those of $G(x, t)$, which are $\tilde{\mu}_k^2$, by the inequalities [8, p. 445]:

$$(33) \quad |\nu_{kn} - \tilde{\mu}_k^2| \leq M_1(\eta_n), \quad k = 1, 2, \dots$$

The ν_{kn} , $k = 1, \dots, n$, are exactly the eigenvalues of the matrix $L_n \equiv (w_{in} G(x_{in}, x_{jn}))$, which is similar to the Hermitian matrix $A^{(n)}$ defined by

$$A_{ij}^{(n)} \equiv \sqrt{w_{in} w_{jn}} G(x_{in}, x_{jn}) = \sqrt{w_{in} w_{jn}} [G_n(x_{in}, x_{jn}) - \eta_n(x_{in}, x_{jn})].$$

Now, a procedure similar to that described in [8] for characteristic values of kernels leads to the inequalities

$$|\tilde{\mu}_{kn}^2 - \nu_{kn}| \leq M_{1n}(\eta_n), \quad k = 1, 2, \dots,$$

where $\tilde{\mu}_{kn} = 0$ for $k > n$, which together with (33) yields

$$|\tilde{\mu}_{kn}^2 - \tilde{\mu}_k^2| \leq M_1(\eta_n) + M_{1n}(\eta_n), \quad k = 1, 2, \dots$$

Corollary 3 follows from (33).

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