

## On Lower and Upper Bounds of the Difference Between the Arithmetic and the Geometric Mean

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**Abstract.** Lower and upper bounds of the difference between the arithmetic and the geometric mean of  $n$  quantities are given here in terms of  $n$ , the smallest value  $a$  and the largest value  $A$  of given  $n$  quantities. Also, an upper bound for the difference, independent of  $n$ , is given in terms of  $a$  and  $A$ . All the bounds obtained are sharp.

**1. Introduction.** Let  $a_1, \dots, a_n$  be  $n$  quantities such that  $0 < a \equiv a_1 \leq a_2 \leq \dots \leq a_n \equiv A$ . Let  $A_n$  be their arithmetic mean,  $G_n$  their geometric mean. Trivial lower and upper bounds of the difference  $A_n - G_n$  are 0 and  $A - a$  respectively. A nice upper bound has been obtained in [2]. Here we shall prove the following inequalities:

$$n^{-1}(\sqrt{A} - \sqrt{a})^2 \leq A_n - G_n \leq ca + (1 - c)A - a^c A^{1-c}$$

where

$$c = \frac{\log[(A/(A - a))\log A/a]}{\log A/a}.$$

The inequalities give lower and upper bounds of the difference  $A_n - G_n$  in terms of the smallest value  $a$  and the largest value  $A$  of the given  $n$  quantities. Instead of a discrete method, a continuous and analytic approach is used to obtain the inequalities.

**2. Lower Bounds.** We consider the lower bound of  $n$  quantities  $0 < a \equiv a_1 \leq \dots \leq a_{k-1} \leq a_{k+1} \leq \dots \leq a_n \equiv A$  with  $a_k \equiv x$ ,  $1 < k < n$ , to be a variable in the interval  $[a, A]$ . Let the arithmetic and the geometric means of the fixed  $n - 1$  quantities be  $A_{n-1}$  and  $G_{n-1}$ , respectively. Then

$$A_n - G_n = n^{-1} \{(n - 1)A_{n-1} + x\} - \{G_{n-1}^{n-1}x\}^{1/n} \equiv D_n(x).$$

Since  $D'_n(x) = 0$  at  $x = G_{n-1}$ , the lower bound of  $D_n(x)$  for  $x$  in the interval  $[a, A]$  is

$$D_n(G_{n-1}) = ((n - 1)/n)(A_{n-1} - G_{n-1}).$$

This result can also be found in [1, p. 12], but the method used here seems to be simpler and more straightforward. By repeating this process, we have

$$\begin{aligned} A_n - G_n &\geq \frac{n-1}{n}(A_{n-1} - G_{n-1}) \geq \frac{n-1}{n} \frac{n-2}{n-1}(A_{n-2} - G_{n-2}) \geq \dots \\ &\geq \frac{2}{n}(A_2 - G_2) = \frac{2}{n} \left( \frac{a+A}{2} - \sqrt{aA} \right) = \frac{1}{n}(\sqrt{A} - \sqrt{a})^2. \end{aligned}$$

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Equality holds only if  $a_2 = \dots = a_{n-1} = \sqrt{a_1 a_n} = \sqrt{aA}$ .

**3. Upper Bounds.** Now we investigate the upper bound of  $A_n - G_n$ . The maximum of  $D_n(x)$  on  $[a, A]$  is attained at the endpoint  $a$  or  $A$ . Thus, the maximum of  $A_n - G_n$  of  $n$  quantities is attained when  $a \equiv a_1 = \dots = a_k \leq a_{k+1} = \dots = a_n \equiv A$  for some  $k, 1 < k < n$ , with the form

$$\frac{kc + (n - k)A}{n} - \{a^k A^{n-k}\}^{1/n} = a \left[ \frac{k + (n - k)A/a}{n} - \{A/a\}^{(n-k)/n} \right].$$

For the sake of simplicity, let  $a = 1$  and consider the function

$$D(x) = \frac{x + (n - x)A}{n} - A^{(n-x)/n}.$$

Through straight calculation, we have  $D'(x) = 0$  for  $x = cn$ , where

$$c = \frac{\log[(A/(A - 1))\log A]}{\log A}.$$

Thus, the upper bound for  $A_n - G_n$  is  $D(cn) = c + (1 - c)A - A^{1-c} \equiv U$  which is independent of  $n$ . By repeated application of L'Hospital's rule, we have

$$\lim_{A \rightarrow 1} c = \frac{1}{2} \quad \text{and} \quad \lim_{A \rightarrow \infty} c = 0.$$

The upper bound is attained only at its limiting case  $A \rightarrow 1$  or  $n \rightarrow \infty$ . But this is the best possible bound independent of the positive integer  $n \geq 2$ . For a fixed  $n$ , the sharp upper bound of  $A_n - G_n$  is attained by  $D(k_n)$  with  $k_n = [cn]$  or  $[cn] + 1$ , where  $[cn]$  denotes the largest integer not greater than  $cn$ . Therefore, we have lower and upper bounds of  $A_n - G_n$ , both dependent and independent of  $n$ , in terms of  $a = 1$  and  $A$  as follows:

$$0 \leq n^{-1}(\sqrt{A} - 1)^2 \leq A_n - G_n \leq D(k_n) \leq c + (1 - c)A - A^{1-c}.$$

Some numerical data are shown below.

TABLE ( $a = 1$ )

$A$	$c$	$k_{10}$	$D(k_{10})$	$U$
1	0.5		0	0
1.001	0.499925	5	$1.25 \times 10^{-7}$	$1.25 \times 10^{-7}$
1.1	0.496029	5	$1.191152 \times 10^{-3}$	$1.191227 \times 10^{-3}$
2	0.471234	5	0.085786	0.086071
$e$	0.458675	5	0.210420	0.211867
5	0.434331	4	0.773472	0.777337
10	0.407973	4	2.418928	2.419591
100	0.333805	3	45.18114	45.45570
$10^5$	0.212238	2	$7.00002 \times 10^4$	$7.00906 \times 10^4$
$10^{10}$	0.136222	1	$8.00000 \times 10^9$	$8.20349 \times 10^9$

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1. E. F. BECKENBACH & R. E. BELLMAN, *Inequalities*, 2nd rev. ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F.*, Band 30, Springer-Verlag, New York, 1965. MR 33 #236.
2. C. LOEWNER & H. B. MANN, "On the difference between the geometric and the arithmetic mean of  $n$  quantities," *Advances in Math.*, v. 5, 1971, pp. 472–473. MR 43 #4982.