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Sums of Distinct Elements from a Fixed Set

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Abstract. A sequence of natural numbers is complete if every large integer is a sum of distinct elements of the sequence. The greatest integer which is not such a sum is called the threshold of completeness. Richert developed a method to compute the threshold of completeness. We prove that Richert's method applies to a large class of complete sequences. Further, we consider in some detail these concepts for the sequences of powers (with fixed exponents) and give numerical results.

1. Let $M = \{m_1, m_2, \ldots\}$ be any increasing sequence of distinct natural numbers. M is called *complete* if every sufficiently large integer may be expressed as a sum of distinct elements of M. If M is complete, then there is a greatest integer which cannot be so expressed. This is called the *threshold of completeness* of M and is denoted by $\theta(M)$.

A survey of papers on complete sequences is given by Graham [7]. The threshold of completeness has been computed for a number of sequences by Sprague [13], Richert [11], [12], Makowski [10], Graham [6], [7], Lin [9], and Dressler, Makowski and Parker [5]. Some of these values have been obtained independently by others (see [1], [2], [3], [4], [8]). The computations have been based on a theorem of Richert [11] or its underlying idea. An algorithm is given by Lin [9]. In this paper we give a partial answer to the question: For which complete sequences may Lin's algorithm be used to compute the threshold of completeness? In particular, we make a closer study of the threshold of completeness of the sequences of powers with fixed exponents.

2. We shall use the following notations:

$(a, b]$ denote the integers n such that $a < n \leqslant b$, we call it an interval and $b - a$ is its length;

$M_{(k)} = \{m_1, m_2, \ldots, m_k\}$;

$N^\alpha = \{[n^\alpha] \mid n = 1, 2, \ldots\}$ where $[x]$ denote the greatest integer $\leqslant x$;

$N_0 = \{0, 1, 2, \ldots\}$;

we define a relation Æ between sequences of integers by: P Æ Q if each element of P may be expressed as a sum of distinct elements of Q.

LEMMA 2.1. *If $m_{i+1} \leqslant 2m_i$ for $i > K$ and*

$$(a, a + m_{k+1}] \text{ Æ } M_{(k)} \tag{2.1}$$

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for some $a \in N_0$ *and some* $k \geqslant K$, *then*

$$(a, a + m_{l+1}] \text{ Æ } M_{(l)} \tag{2.2}$$

for all $l \geqslant k$.

We prove (2.2) by induction on l. By (2.1) it is true for $l = k$. Suppose it is true for some $l \geqslant k$. Adding m_{l+1} to each element of $(a, a + m_{l+1}]$ and merging the two intervals, we get $(a, a + 2m_{l+1}]$ Æ $M_{(l+1)}$. Since $m_{l+2} \leqslant 2m_{l+1}$, this completes the induction.

THEOREM 2.1 (RICHERT [11]). *If* $m_{i+1} \leqslant 2m_i$ *for* $i > K$ *and* $(a, a + m_{k+1}]$ Æ $M_{(k)}$ *for some* $a \in N_0$ *and some* $k \geqslant K$, *then* M *is complete and* $\theta(M) \leqslant a$.

Proof. Let $b > a$. Since $m_l \longrightarrow \infty$ for $l \longrightarrow \infty$, $b \leqslant a + m_{l+1}$ for some l. By Lemma 2.1 $\{b\}$ Æ $M_{(l)}$. Hence $\{b\}$ Æ M.

Theorem 2.1 provides a method to calculate $\theta(M)$. A discussion of an algorithm is given by Lin [9]. The crucial point is the existence of a and k such that (2.1) is satisfied. This will be further discussed in Lemma 2.2 and Theorem 2.3 below. If we want to compute $\theta_n = \theta(\{m_n, m_{n+1}, \ldots\})$ for $n = 1, 2, \ldots, l$, the algorithm may be modified to yield all these values in one run (see Kløve [8]). We give a sketch of this modified algorithm.

Algorithm θ.

Step 1. $k := l - 1$; $N_{l-1} = \{0\}$;
Step 2. $k := k + 1$; $N_k = N_{k-1} \cup \{n + m_k \mid n \in N_{k-1}\}$;
Step 3. *If* $k < K$, *then go to* Step 2.
Step 4. *If* N_k does not contain an interval of length $\geqslant m_{k+1}$, *then go to* Step 2;
Step 5. $a :=$ least x such that $(x, x + m_{k+1}] \subseteq N_k$;
Step 6. $k := k + 1$; *if* $m_k > a$, *then go to* Step 8;
Step 7. $N_k := N_{k-1} \cup \{n + m_k \mid n \in N_{k-1} \;\&\; n \leqslant a - m_k\}$;
$a :=$ least x such that $(x, a + 1] \subseteq N_k$; *go to* Step 6;
Step 8. $\theta_l := a$;
Step 9. $k := l$;
Step 10. *If* $k > 1$, *then* $k := k - 1$, *else stop.*
Step 11. $N_k := N_{k+1} \cup \{n + m_k \mid n \in N_{k+1} \;\&\; n \leqslant \theta_{k+1} - m_{k+1}\}$;
$\theta_k :=$ least x such that $(x, \theta_{k+1} + 1] \subseteq N_k$; *go to* Step 10.

The actual algorithm will depend on how we choose to represent the sets N_k. Clearly, for $l = 1$ the algorithm will give just $\theta(M)$.

The following theorem may be used to simplify the algorithm (this simplified algorithm will usually be less efficient, however).

THEOREM 2.2. *If* $m_{i+1} \leqslant 2m_i$ *for* $i > K$ *and*

$$(a, a + m_{k+1}] \text{ Æ } M_{(k)} \tag{2.3}$$

for some $a \in N_0$ *and some* $k \geqslant K$, *then*

$$(\theta(M), \theta(M) + m_{l+1}] \text{ Æ } M_{(l)} \quad \textit{for some } l \geqslant K.$$

Proof. Let $a^* = \min\{a \mid (2.3)$ is satisfied for some $k \geqslant K\}$. By Theorem 1, $a^* \geqslant \theta(M)$. Suppose $a^* > \theta(M)$. Then $a^* = \Sigma_j m_{i_j}$ where $i_j \leqslant l_1$, for some l_1. By assumption

$$(a^*, a^* + m_{L+1}] \text{ Æ } M_{(L)}$$

for some $L \geqslant K$. Let $l = \max(l_1, L)$. Then, by Lemma 2.1 $(a^*, a^* + m_{l+1}] \text{ Æ } M_{(l)}$. Further $\{a^*\} \text{ Æ } M_{(l)}$. Hence

$$(a^* - 1, a^* - 1 + m_{l+1}] \text{ Æ } M_{(l)}$$

contradicting the definition of a^*.

We now turn to the question: For which complete sequences may the algorithm above be used to compute the threshold of completeness? A partial answer is given by the following lemma and Theorem 2.3 below.

LEMMA 2.2. *If M is complete and $m_{k+1} \leqslant 2m_k - \theta(M) - 1$ for $k \geqslant K$, let L be the minimum $l \geqslant K$ such that $m_{l+1} \geqslant 1 + \theta(M) + m_k$. Then for all $l \geqslant L$*

$$(\theta(M), \theta(M) + m_{l+1}] \text{ Æ } M_{(l)}.$$

Proof. Let $\theta = \theta(M)$. We may assume that $m_K \geqslant \theta - 2$ (otherwise we just increase K). First, we prove by induction on n that

$$(\theta, m_{n+1} - 1] \cup (\theta + m_K, 2m_n - 1] \text{ Æ } M_{(n)} \tag{2.4}$$

for $n \geqslant K$. By assumption $(\theta, m_{n+1} - 1] \text{ Æ } M$ and since m_k for $k \geqslant n + 1$ may never occur as a summand, we get

$$(\theta, m_{n+1} - 1] \text{ Æ } M_{(n)} \tag{2.5}$$

for all n. In particular, (2.5) is true for $n = K - 1$. Adding m_K to each element,we get $(\theta + m_K, 2m_K - 1] \text{ Æ } M_{(K)}$, and so (2.4) is true for $n = K$. Suppose it is true for some $n \geqslant K$. In particular,

$$(\theta + m_K, 2m_n - 1] \text{ Æ } M_{(n)}. \tag{2.6}$$

Adding m_{n+1} to each element of the set in (2.5), we get

$$(\theta + m_{n+1}, 2m_{n+1} - 1] \text{ Æ } M_{(n+1)}. \tag{2.7}$$

Since $\theta + m_{n+1} < 2m_n$, (2.5), (2.6), and (2.7) show that (2.4) is true for n replaced by $n + 1$ and the induction is complete. Now choose $l \geqslant K$ such that $m_{l+1} \geqslant \theta + 1 + m_K$. Then $(\theta, 2m_l - 1] \text{ Æ } M_{(l)}$. Finally,

$$2m_l - 1 = \theta + (2m_l - 1 - \theta) \geqslant \theta + m_{l+1}$$

and the lemma is proved.

THEOREM 2.3. *Let M be complete. If, for some $\epsilon > 0$, $m_{l+1} \leqslant (2 - \epsilon)m_l$ for all $l >> 0$, then $\theta(M)$ may be computed by algorithm θ.*

Proof. Choose $k > K$ such that $\epsilon m_k \geqslant \theta(M) + 1$. Then

$$m_{k+1} \leqslant 2m_k - \epsilon m_k \leqslant 2m_k - \theta(M) - 1.$$

By Lemma 2.2 there exists an l such that

$$(\theta(M), \theta(M) + m_{l+1}] \text{ Æ } M_{(l)}$$

and algorithm θ applies.

We now give a trivial lemma which may be used to find a lower bound for $\theta(M)$.

LEMMA 2.3. *If M is complete, then*

$$\theta(M) \geqslant \max_n (m_{n+1} - 2^n - 1).$$

Proof. The number of distinct sums of distinct elements of $M_{(n)}$ is $\leqslant 2^n$. Hence, if $2^n + 1 \leqslant m_{n+1}$, then there is at least one integer in $[m_{n+1} - 2^n - 1, m_{n+1})$ which cannot be expressed as a sum of distinct elements of M. Hence $\theta(M) \geqslant m_{n+1} - 2^n - 1$.

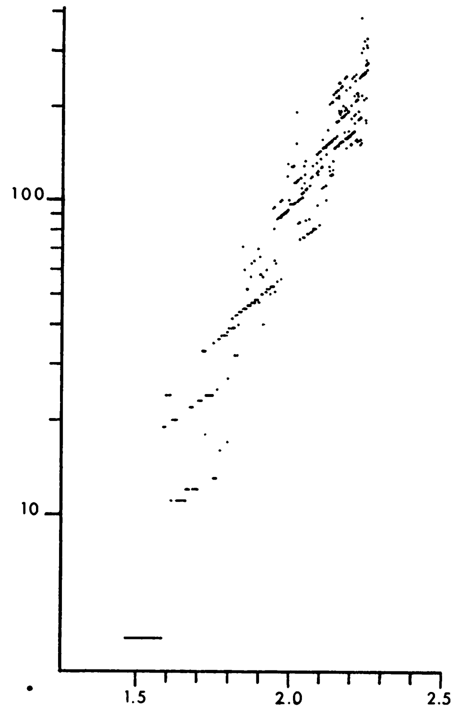

FIGURE 1. $t(\alpha)$ *for* $1.50 \leqslant \alpha \leqslant 2.25$.

3. Sprague [13], [14] proved that N^α is complete for $\alpha = 2, 3, \ldots$ and found that $\theta(N^2) = 128$. Since N^n is a subsequence of $N^{n/m}$, N^α is complete for all rational α. Graham [6] proved that $\theta(N^3) = 12758$ and Lin [9] that $\theta(N^4) = 5134240$. Let $t(\alpha) = \theta(N^\alpha)$. The rest of this paper is concerned with $t(\alpha)$.

THEOREM 3.1. *If N^β is complete, then there exists a $\beta' > \beta$ such that $t(\alpha)$ is constant on* $[\beta, \beta')$.

Note that $[\beta, \beta')$ denotes an interval in the ordinary sense.

Proof. Clearly $[(k+1)^\beta] < 2[k^\beta] - t(\beta) - 1$ for $k >> 0$. Hence, by Lemma 2.2, there exists an l such that

$$(t(\beta), t(\beta) + [(l+1)^\beta]] \text{ Æ } N^\beta_{(l)}. \tag{3.1}$$

By Lemma 2.1 we may assume that

$$[l^\beta] > t(\beta). \tag{3.2}$$

TABLE 1

Values of $t(0.1\beta + 0.01\gamma)$

$\beta \backslash \gamma$	0	2	4	6	8
15	4	4	4	4	4
16	24	20	11	11	22
17	23	33	24	13	37
18	39	43	44	46	64
19	58	51	94	88	91
20	128	115	76	111	81
21	127	169	147	193	159
22	166	183	259	221	294
23	306	310	306	517	409
24	530	646	728	493	1095
25	724	702	975	798	892
26	1039	1180	1583	1324	1671
27	1702	1741	1791	1925	2286
28	1987	2873	2671	3909	4711
29	3821	4882	4312	4866	5814
30	12758	7110	6206	8895	8536
31	13399	10594	11530	11008	16458
32	11335	18114	18654	18165	21637
33	21090	24770	23807	28420	29178
34	32666	40701	40018	46056	49473
35	54714	54869	56832	65719	86669
36	74648	95679	88685	94399	109647
37	132511	126425	158215	177256	174059
38	206162	213823	240047	247208	286548
39	324446	331169	365194	395838	403710
40	5134240	483960	635701	610633	726860
41	742867	831254	812774	984068	981046

Choose $\beta' > \beta$ such that $[n^{\alpha}]$ is constant on $[\beta, \beta')$ as a function of α for $n = 1, 2, \ldots, l+1$, e.g.

(3.3) $$\beta' = \min\{\log([n^{\beta}] + 1)/\log n \mid 1 < n \leqslant l+1\}.$$

Then, for all $\alpha \in [\beta, \beta')$,

$$(t(\beta), t(\beta) + [(l+1)^{\alpha}]] \text{ Æ } N^{\alpha}_{(l)}.$$

By Theorem 2.1 $t(\alpha) \leqslant t(\beta)$. Suppose $t(\alpha) < t(\beta)$ for some $\alpha \in [\beta, \beta')$. Then

$$t(\beta) = \sum_j [n_j^{\alpha}] = \sum_j [n_j^{\beta}]$$

since $n_j < l$ for all j $([n_j^{\alpha}] \leqslant t(\beta) < [l^{\beta}] \leqslant [l^{\alpha}])$, a contradiction. Hence $t(\alpha) = t(\beta)$ for all $\alpha \in [\beta, \beta')$.

We see that, given β, we may compute a β' satisfying the theorem as follows: Using algorithm θ, we compute an l satisfying (3.1) and (3.2). Then a β' is given by (3.3). Further, we may replace β by β' and compute a new interval. Using this algorithm, we computed $t(\alpha)$ for $1 \leqslant \alpha \leqslant \log 686/\log 15 \approx 2.412$. In particular,

$$t(\alpha) = 0 \quad \text{for } 1 \leqslant \alpha < \log 5/\log 3 \approx 1.465,$$

$$t(\alpha) = 4 \quad \text{for } \log 5/\log 3 \leqslant \alpha < \log 3/\log 2 \approx 1.585.$$

A plot of $t(\alpha)$ versus α for $1.50 \leqslant \alpha \leqslant 2.25$ is given in Fig. 1.

We have computed $t(\alpha)$ for $\alpha = 1.50\ (0.02)\ 4.18$ using algorithm θ. The values are given in Table 1. (The value for $t(4)$ is taken from Lin [9] and has not been recomputed.) It is striking that the value of $t(\alpha)$ at $\alpha = 4$ is much greater than the surrounding values. The referee suggests that this is probably due to the fact that $x^4 \equiv 0, 1 \pmod{16}$.

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