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Properties of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Functions

By Alois Schett

Abstract. Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions and tables for the first fifteen leading terms are given. Relations of these coefficients with the randomization distributions are shown.

Little is known about the Taylor series expansion coefficients of the Jacobian elliptic functions $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$. No recurrence formula exists for these coefficients. Only four to five leading terms of the series are given in literature ([1], [2]).

We present in this paper properties of these coefficients, show relations between them and randomization distributions [3, p. 51], and give tables for the first fifteen leading terms.

We consider the differential equations

$$\begin{aligned} &\frac{d}{du} y_1(u) - C_1 y_2(u) y_3(u) = 0, \\ (1) \qquad &\frac{d}{du} y_2(u) - C_2 y_1(u) y_3(u) = 0, \\ &\frac{d}{du} y_3(u) - C_3 y_1(u) y_2(u) = 0. \end{aligned}$$

Solution functions of (1) for $C_1 = 1$, $C_2 = -1$, $C_3 = -k^2$ are the Jacobian elliptic functions $y_1 = \mathrm{sn}(u, k)$, $y_2 = \mathrm{cn}(u, k)$, $y_3 = \mathrm{dn}(u, k)$ ([1], [2]).

The formal Taylor series of the functions y_1, y_2, y_3 read

$$\begin{aligned} y_1(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum a_{j_1 j_2 j_3} C_1^{j_1} C_2^{j_2} C_3^{j_3} y_{10}^{i_1} y_{20}^{i_2} y_{30}^{i_3} \right], \\ (2) \qquad y_2(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum b_{h_1 h_2 h_3} C_1^{h_1} C_2^{h_2} C_3^{h_3} y_{10}^{s_1} y_{20}^{s_2} y_{30}^{s_3} \right], \\ y_3(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[\sum c_{r_1 r_2 r_3} C_1^{r_1} C_2^{r_2} C_3^{r_3} y_{10}^{q_1} y_{20}^{q_2} y_{30}^{q_3} \right]. \end{aligned}$$

The summation over the indices j_1, j_2, j_3; h_1, h_2, h_3; r_1, r_2, r_3 and their relation to the exponents of y_{10}, y_{20}, y_{30} are specified in Theorem I.

$$y_{m0} = y_m(u_0) \qquad (m = 1, 2, 3).$$

For $u_0 = 0$, $y_{10} = 0$, $y_{20} = y_{30} = 1$, this series is convergent in the region $|u| < K'$ [2], where

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$$K' = K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - k'^2 \sin^2\theta)}}, \qquad k' = \sqrt{1 - k^2}.$$

For the explicit series of (2), the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$, $c_{r_1 r_2 r_3}$ have to be determined and summation over the indices has to be specified.

THEOREM I. *$a_{j_1 j_2 j_3} \neq 0$ only for*

$$j_1 = 1, 2, \dots, J_1; \qquad J_1 = \begin{cases} n/2 & \text{for } n \text{ even,} \\ (n+1)/2 & \text{for } n \text{ odd,} \end{cases}$$

$$\begin{matrix} j_2 = 0, 1, \dots, J_2; \\ j_3 = 0, 1, \dots, J_3; \end{matrix} \qquad J_2, J_3 = \begin{cases} n/2 & \text{for } n \text{ even,} \\ (n-1)/2 & \text{for } n \text{ odd,} \end{cases}$$

and j_1, j_2, j_3 satisfying the relation $j_1 + j_2 + j_3 = n$. i_1, i_2, i_3 is obtained from the relations $i_1 = n + 1 - 2j_1$, $i_2 = n - 2j_2$, $i_3 = n - 2j_3$. $b_{h_1 h_2 h_3} \neq 0$ only for $h_1 = j_3$, $h_2 = j_1$, $h_3 = j_2$.

The exponents s_1, s_2, s_3 are related to h_1, h_2, h_3 as follows: $s_1 = n - 2h_1$, $s_2 = n + 1 - 2h_2$, $s_3 = n - 2h_3$. $c_{r_1 r_2 r_3} \neq 0$ only for $r_1 = j_2$, $r_2 = j_3$, $r_3 = j_1$. The exponents q_1, q_2, q_3 are defined by the relations $q_1 = n - 2r_1$, $q_2 = n - 2r_2$, $q_3 = n + 1 - 2r_3$.

For $u_0 = 0$, i.e.,

$$y_1(u_0 = 0) = \mathrm{sn}(0, k) = 0, \quad y_2(u_0 = 0) = \mathrm{cn}(0, k) = 1,$$

$$y_3(u_0 = 0) = \mathrm{dn}(0, k) = 1,$$

we obtain

$$a_{j_1 j_2 j_3} \neq 0 \quad \text{only for } j_1 = (n+1)/2; j_2 = 0, 1, \dots, (n-1)/2; n \text{ odd},$$

$$b_{h_1 h_2 h_3} \neq 0 \quad \text{only for } h_3 = n/2; h_1 = 0, 1, \dots, n/2 - 1; n \text{ even},$$

$$c_{r_1 r_2 r_3} \neq 0 \quad \text{only for } r_2 = n/2; r_3 = 1, 2, \dots, n/2; n \text{ even}.$$

In the following table the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$, $c_{r_1 r_2 r_3}$ and the exponents $i_1, i_2, i_3; s_1, s_2, s_3; q_1, q_2, q_3$ are given as an illustration for $n = 3$ and $n = 4$.

TABLE I

	$a_{j_1 j_2 j_3}$	j_1	j_2	j_3	i_1	i_2	i_3	$b_{h_1 h_2 h_3}$	h_1	h_2	h_3	s_1	s_2	s_3	$c_{r_1 r_2 r_3}$	r_1	r_2	r_3	q_1	q_2	q_3
$n=3$	1	2	1	0	0	1	3	4	1	1	1	1	2	1	4	1	1	1	1	1	2
	1	2	0	1	0	3	1	1	1	2	0	1	0	3	1	1	0	2	1	3	0
	4	1	1	1	2	1	1	1	0	2	1	3	0	1	1	0	1	2	3	1	0
$n=4$	14	2	1	1	1	2	2	4	2	1	1	0	3	2	4	2	1	1	0	2	3
	1	2	2	0	1	0	4	1	2	2	0	0	1	4	1	2	0	2	0	4	1
	4	1	2	1	3	0	2	14	1	2	1	2	1	2	14	1	1	2	2	2	1
	1	2	0	2	1	4	0	4	1	1	2	2	3	0	4	1	2	1	2	0	3
	4	1	1	2	3	2	0	1	0	2	2	4	1	0	1	0	2	2	4	0	1

The following identities and symmetries are valid:

THEOREM II.

$$a_{j_1 j_2 j_3} = b_{j_3 j_1 j_2} = c_{j_2 j_3 j_1},$$

$$a_{j_1 j_2 j_3} = a_{j_1 j_3 j_2};$$

and therefore,

$$b_{j_3 j_1 j_2} = b_{j_2 j_1 j_3},$$

$$c_{j_2 j_3 j_1} = c_{j_3 j_2 j_1}.$$

Theorems III/1, III/2 and III/3 show relations between the randomization distribution and the elements $a_{j_1 j_2 j_3}$, $b_{h_1 h_2 h_3}$ and $c_{r_1 r_2 r_3}$.

THEOREM III/1. *The sum of all elements $a_{j_1 j_2 j_3}$ for a given n is equal to $n!$.*

Example: $n = 4$.

$$a_{112} + a_{121} + a_{202} + a_{211} + a_{220} = 4 + 4 + 1 + 14 + 1 = 24 = 4!$$

THEOREM III/2. *For a given $n = 2$, the following relation is valid:*

$$\sum_{j_3=0}^{J_3} a_{j_1 j_2 j_3} \geqslant RU_{j_2}, \qquad j_2 = 0, 1, \ldots, J_2,$$

where RU_{j_2} is the number of permutations of n natural numbers with j_2 runs up. Tables of the numbers RU_{j_2} are given in [3, *p.* 260, *Table* 7.2.2].

Example: $n = 3$.

$$a_{201} = 1 = RU_0, \qquad a_{210} + a_{111} = 1 + 4 = 5 = RU_1.$$

	Permutations	One run up (underlined)	Zero run up (underlined)
	123	$\underline{123}$	123
	132	$\underline{13}2$	132
	231	$\underline{23}1$	231
	213	$2\underline{13}$	213
	312	$3\underline{12}$	312
	321	321	$\underline{321}$
Total	$6 = n!$	$5 = RU_1$	$1 = RU_0$

THEOREM III/3. *For a given $n \geqslant 2$ the following relation is valid:*

$$\sum_{j_2=0}^{J_2} a_{j_1 j_2 j_3} = P_{j_1}, \qquad j_1 = 1, 2, \ldots, J_1,$$

where P_{j_1} is the number of permutations of n natural numbers with $j_1 - 1$ peaks. The *numbers* P_{j_1} *are tabulated in* [3, *p.* 261, *Table* 7.3].

Example: $n = 3$.

$$a_{111} = 4 = P_1, \qquad a_{201} + a_{210} = 1 + 1 = 2 = P_2.$$

	Permutations	One peak (underlined)	Zero peak (underlined)
	123	123	$\underline{123}$
	132	$\underline{132}$	132
	231	$\underline{231}$	231
	213	213	$\underline{213}$
	312	312	$\underline{312}$
	321	321	$\underline{321}$
Total	$6 = n!$	$2 = P_2$	$4 = P_1$

Similar results can be obtained for $b_{h_1h_2h_3}$ and $c_{r_1r_2r_3}$ using Theorem II.

These theorems can be proved by mathematical induction. Theorem III/1 follows from Theorem III/2 or Theorem III/3 since the number of permuations of n natural numbers is equal to $n!$.

Table II, in the microfiche section attached to this issue, lists the elements $a_{j_1j_2j_3}$ for $n = 0, 1, 2, \ldots, 15$.

Putting $u_0 = 0$ the explicit terms of the series for sn(u, k), cn(u, k) and dn(u, k) read (Theorem I, Theorem II and Table II):

$$\begin{aligned}
\mathrm{sn}(u, k) = {} & u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - (1 + 135k^2 + 135k^4 + k^6)\frac{u^7}{7!} \\
& + (1 + 1228k^2 + 5478k^4 + 1228k^6 + k^8)\frac{u^9}{9!} \\
& - (1 + 11069k^2 + 165826k^4 + 165826k^6 + 11069k^8 + k^{10})\frac{u^{11}}{11!} \\
& + (1 + 99642k^2 + 4494351k^4 + 13180268k^6 + 4494351k^8 \\
& \qquad + 99642k^{10} + k^{12})\frac{u^{13}}{13!} \\
& - (1 + 896803k^2 + 116294673k^4 + 834687179k^6 + 834687179k^8 \\
& \qquad + 116294673k^{10} + 896803k^{12} + k^{14})\frac{u^{15}}{15!} + \cdots \\
= {} & \sum_{n_0 = 1;\,(n_0 \text{ odd})}^{\infty} (-1)^{(n_0-1)/2} \left(\sum_{j_2=0}^{(n_0-1)/2} a_{j_1j_2j_3} k^{2j_2} \right) \frac{u^{n_0}}{n_0!}, \quad j_1 = \frac{n_0 + 1}{2}
\end{aligned}$$

(terms for $n_0 \leqslant 7$ are given in [2]);

$$\begin{aligned}
\mathrm{cn}(u, k) = 1 &- \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!} \\
&+ (1 + 408k^2 + 912k^4 + 64k^6)\frac{u^8}{8!} \\
&- (1 + 3688k^2 + 30768k^4 + 15808k^6 + 256k^8)\frac{u^{10}}{10!} \\
&+ (1 + 33212k^2 + 870640k^4 + 1538560k^6 + 259328k^8 + 1024k^{10})\frac{u^{12}}{12!} \\
&- (1 + 298932k^2 + 22945056k^4 + 106923008k^6 + 65008896k^8 \\
&\qquad + 4180992k^{10} + 4096k^{12})\frac{u^{14}}{14!} + \cdots \\
= 1 &+ \sum_{n_e=2;\,(n_e\ \text{even})}^{\infty} (-1)^{n_e/2}\left(\sum_{h_1=0}^{n_e/2-1} b_{h_1h_2h_3}k^{2h_1}\right)\frac{u^{n_e}}{n_e!}, \qquad h_3 = n_e/2
\end{aligned}$$

(terms for $n_e \leqslant 8$ are given in [2]);

$$\begin{aligned}
\mathrm{dn}(u, k) = 1 &- k^2\frac{u^2}{2!} + (4 + k^2)k^2\frac{u^4}{4!} - (16 + 44k^2 + k^4)k^2\frac{u^6}{6!} \\
&+ (64 + 912k^2 + 408k^4 + k^6)k^2\frac{u^8}{8!} \\
&- (256 + 15808k^2 + 30768k^4 + 3688k^6 + k^8)\frac{u^{10}k^2}{10!} \\
&+ (1024 + 259328k^2 + 1538560k^4 + 870640k^6 + 33212k^8 + k^{10})\frac{u^{12}k^2}{12!} \\
&- (4096 + 4180992k^2 + 65008896k^4 + 106923008k^6 \\
&\qquad + 22945056k^8 + 298932k^{10} + k^{12})\frac{u^{14}k^2}{14!} + \cdots \\
= 1 &+ \sum_{n_e=2;\,(n_e\ \text{even})}^{\infty} (-1)^{n_e/2}\left(\sum_{r_3=1}^{n_e/2} c_{r_1r_2r_3}k^{2(r_3-1)}\right)\frac{u^{n_e}k^2}{n_e!}, \qquad r_2 = n_e/2
\end{aligned}$$

(terms for $n_e \leqslant 8$ are given in [2]).

CEN
Saclay, France

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