A Quadratically Convergent Iteration Method for Computing Zeros of Operators Satisfying Autonomous Differential Equations

By L. B. Rall*

Abstract. If the Fréchet derivative $P'$ of the operator $P$ in a Banach space $X$ is Lipschitz continuous, satisfies an autonomous differential equation $P'(x) = f(P(x))$, and $f(0)$ has the bounded inverse $\Gamma$, then the iteration process

$$x_{n+1} = x_n - \Gamma P(x_n), \quad n = 0, 1, 2, \ldots,$$

is shown to be locally quadratically convergent to solutions $x = x^*$ of the equation $P(x) = 0$. If $f$ is Lipschitz continuous and $\Gamma$ exists, then the global existence of $x^*$ is shown to follow if $P(x)$ is uniformly bounded by a sufficiently small constant. The replacement of the uniform boundedness of $P$ by Lipschitz continuity gives a semilocal theorem for the existence of $x^*$ and the quadratic convergence of the sequence $\{x_n\}$ to $x^*$.

Successive approximations $x_1, x_2, \ldots$ to a solution $x = x^*$ of the operator equation $P(x) = 0$ in a Banach space $X$ can be obtained under suitable conditions from an iteration process of the form

$$x_{n+1} = x_n - [P'(y_n)]^{-1} P(x_n), \quad n = 0, 1, 2, \ldots,$$

where the initial approximation $x_0$ and the sequence $\{y_n\}$ are given, and the existence of the inverses of the (Fréchet) derivatives $\{P'(y_n)\}$ and the convergence of the sequence $\{x_n\}$ to $x^*$ can be guaranteed. Special cases of (1) are Newton's method ($y_n = x_n$) and the modified Newton's method ($y_n = x_0$); so methods of this type may be characterized as variants of Newton's method, or Newton-like methods ([2], [3]).

1. Local Convergence. It will be assumed that $P(x^*) = 0$ and $\|P'(x) - P'(y)\| \leq K\|x - y\|$, at least in a sufficiently large region containing $x^*$. The inequality [4]

$$\|x_{n+1} - x^*\| \leq \frac{1}{2} K \|P'(y_n)\|^{-1} \|\|x_n - y_n\| + \|y_n - x^*\|\|x_n - x^*\|$$

is useful for estimating the rate of convergence of $\{x_n\}$ to $x^*$. If one takes $y_n = \lambda_n x_n + (1 - \lambda_n) x^*$, $0 \leq \lambda_n \leq 1$, then $\|x_n - y_n\| + \|y_n - x^*\| = \|x_n - x^*\|$, and one has

$$\|x_{n+1} - x^*\| \leq \frac{1}{2} K \|P'(y_n)\|^{-1} \|x_n - x^*\|^2,$$

which shows that convergence will be quadratic if the inverses $[P'(y_n)]^{-1}$ are uniformly

Received June 20, 1975.

AMS (MOS) subject classifications (1970). Primary 65J05.

Key words and phrases. Nonlinear operator equations, iteration methods, quadratic convergence, variants of Newton's method.

*Sponsored by the U.S. Army under Contract No. DAHC04-75-C-0024.

Copyright © 1976, American Mathematical Society
bounded. The method of present interest is obtained by taking $\lambda_n = 0$, so that $y_n = x^*$. If now $\Gamma = [P'(x^*)]^{-1}$ exists and $\|\Gamma\| \leq B^*$, then the iteration process

$$ x_{n+1} = x_n - \Gamma P(x_n), \quad n = 0, 1, 2, \ldots, $$

will be quadratically convergent, with

$$ \|x_{n+1} - x^*\| \leq \frac{1}{2} KB^* \|x_n - x^*\|^2. $$

The iteration process (4) has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, as the operator $\Gamma$ is calculated once and for all. This method can be realized for operators $P$ which satisfy an autonomous differential equation

$$ P'(x) = f(P(x)), $$
as $P'(x^*) = f(0)$ can be evaluated without knowing the value of $x^*$. With the above assumptions one has the following result.

**Theorem 1.** If $\Gamma = [f(0)]^{-1}$ exists, $\|\Gamma\| \leq B^*$, and $x_0$ is such that

$$ \theta = \frac{1}{2} KB^* \|x_0 - x^*\| < 1, $$
then the sequence $\{x_n\}$ defined by (4) converges to $x^*$, with

$$ \|x_n - x^*\| \leq \theta^n \|x_0 - x^*\|, \quad n = 1, 2, \ldots. $$

**Proof.** Inequality (8) follows from (5) and (7) by mathematical induction.

For example, the iteration process

$$ x_{n+1} = x_n - \frac{1}{N}(e^{x_n} - N), \quad n = 0, 1, 2, \ldots, $$

converges quadratically to the solution $x^* = \ln N$ of $P(x) \equiv e^x - N = 0$ for sufficiently close initial approximations $x_0$; here (6) is $P'(x) = P(x) + N$.

2. A Global Existence Theorem. It will be assumed that $\Gamma = [f(0)]^{-1}$ exists, $\|\Gamma\| \leq B^*$, and conditions for the existence of $x^*$ will be obtained.

**Theorem 2.** If $f$ is Lipschitz continuous with constant $\alpha$, $\|P(x)\| \leq \beta$, and

$$ \rho = \alpha \beta B^* < 1, $$
then the equation $P(x) = 0$ has a unique solution $x^*$ to which the sequence $\{x_n\}$ defined by (4) converges, with

$$ \|x^* - x_n\| \leq \frac{\rho^n}{1 - \rho} \|x_1 - x_0\|, \quad n = 0, 1, 2, \ldots. $$

**Proof.** The iteration process (4) may be written as $x_{n+1} = \Gamma F(x_n)$, $n = 0, 1, 2, \ldots$, where $F(x) = f(0) x - P(x)$. From

$$ F'(x) = f(0) - P'(x) = f(0) - f(P(x)) $$
and the Lipschitz continuity of $f$, it follows that

$$ \|F'(x)\| \leq \alpha \|P(x)\|, $$
and the theorem follows from (10) and the contraction mapping principle [3].

If \( P' \) is Lipschitz continuous in a neighborhood of \( x^* \), then the convergence of the sequence \( \{x_n\} \) will be quadratic within this neighborhood as soon as inequality (7) holds with \( x_0 \) replaced by an iterate \( x_n \) sufficiently close to \( x^* \).

3. A Semilocal Existence Theorem. If \( f \) and \( P \) are Lipschitz continuous with constants \( \alpha \) and \( \gamma \), respectively, then it follows from (6) that \( P' \) is Lipschitz continuous with constant \( K = \alpha\gamma \). Furthermore,

\[
\|P(x)\| \leq \|P(x_0)\| + \gamma \|x - x_0\|.
\]

For \( r = \|x - x_0\| \), define

\[
\rho(r) = \alpha B^* \|P(x_0)\| + B^* K r.
\]

If \( \rho(0) = \alpha B^* \|P(x_0)\| < 1 \), then inequality (10) and the contraction mapping principle [3, pp. 84–85] give the following result.

**Theorem 3. If**

\[
\Delta = (1 - \alpha B^* \|P(x_0)\|)^2 - 4 B^* K \|x_1 - x_0\| \geq 0,
\]

then a solution \( x^* \) of the equation \( P(x) = 0 \) exists in the closed ball

\[
V = \left\{ x : \|x - x_0\| \leq \frac{1 - \alpha B^* \|P(x_0)\| - \sqrt{\Delta}}{2 B^* K} = r^* \right\},
\]

and is unique in the open ball

\[
U = \left\{ x : \|x - x_0\| < \frac{1 - \alpha B^* \|P(x_0)\|}{B^* K} \right\}.
\]

By itself, the contraction mapping principle only guarantees that

\[
\|x_n - x^*\| \leq (\rho^*)^n r^*, \quad n = 0, 1, 2, \ldots,
\]

where

\[
\rho^* = \rho(r^*) = \frac{1}{2} (1 + \alpha B^* \|P(x_0)\| - \sqrt{\Delta}).
\]

By Theorem 1, however, the convergence of the sequence \( \{x_n\} \) to \( x^* \) will be quadratic for \( n = N, N + 1, \ldots \), where \( N \) is the smallest nonnegative integer satisfying the inequality

\[
\theta = \frac{1}{2} K B^* (\rho^*)^N r^* < 1.
\]