MATHEMATICS OF COMPUTATION, VOLUME 30, NUMBER 134
APRIL 1976, PAGES 312–323

A Computational Technique for Determining the Class Number of a Pure Cubic Field

By Pierre Barrucand, H. C. Williams and L. Baniuk

Abstract. Two different computational techniques for determining the class number of a pure cubic field are discussed. These techniques were implemented on an IBM/370-158 computer, and the class number for each pure cubic field $Q(D^{1/3})$ for $D = 2, 3, \ldots, 9999$ was obtained. Several tables are presented which summarize the results of these computations. Some theorems concerning the class group structure of pure cubic fields are also given. The paper closes with some conjectures which were inspired by the computer results.

1. Introduction. The theory of pure cubic fields $Q(D^{1/3})$, D rational, was founded in 1892 by Markov [12]; in his paper he gives some class numbers and fundamental units, not always in an explicit form. In [7] Dedekind describes a method for determining the class number of a pure cubic field $Q(D^{1/3})$. He also gives a short table of class numbers for some small values of D. Cohn [6] implemented Dedekind's method on a computer and obtained class numbers for some fields in which he could easily determine the regulator. Cohn's technique was modified somewhat by Beach, Williams, and Zarnke in [4], and class numbers were obtained for $Q(D^{1/3})$ for $D = 2, 3, \ldots, 999$. Other tables of class numbers have been calculated by hand by Cassels [5] and Selmer [14]. It should also be mentioned that Angell [1] has recently given a list of class numbers for all cubic fields with negative discriminant greater than $-20,000$.

The purpose of this paper is to present a new technique for determining the class number of $Q(D^{1/3})$. This method is much faster than the computational technique of [6] and [4]. The algorithm was implemented on a computer and the class numbers for $Q(D^{1/3})$ obtained for $D = 2, 3, \ldots, 9999$. The total number of these fields is 8122. We also describe here some of the results of these calculations.

2. Some Properties of Pure Cubic Fields. Let K be any cubic field with discriminant Δ. If $\zeta_K(s)$ is the Riemann zeta function in K, we have

$$\Phi(s) = \zeta_K(s)/\zeta(s) = \sum_{j=1}^{\infty} \alpha(j) j^{-s}.$$

Here $\alpha(j) = \Sigma_{d \mid j} \mu(d) F(j/d)$, where $F(n)$ is the number of distinct ideals of norm n in K. Also, $CRh = \Phi(1)$, where h is the class number, R is the regulator and C is a constant. If $\Delta < 0$, $C = 2\pi/\sqrt{|\Delta|}$ and $R = \log \epsilon_1$, where ϵ_1 (> 1) is the fundamental unit of K.

Received July 28, 1975; revised October 7, 1975.
AMS (MOS) subject classifications (1970). Primary 12A50, 12A30; Secondary 12–04, 12A70.

If D is an integer which is not a perfect cube and $K = Q(D^{1/3})$, the cubic field formed by adjoining $D^{1/3}$ to the rationals, we call K a pure cubic field. Let D be a cube free integer and let $D = ab^2$, where a, b are square free. We have $\Delta = -3k^2$, where

$$k = \begin{cases} 3ab & \text{when } a^2 \not\equiv b^2 \pmod 9, \\ ab & \text{when } a^2 \equiv b^2 \pmod 9. \end{cases}$$

If $a^2 \not\equiv b^2 \pmod 9$, we say (after Dedekind) that K is of type 1; otherwise, we say that K is of type 2.

In K, $\alpha(j)$ is a multiplicative function with $\alpha(1) = 1$;

$$\alpha(3^n) = \begin{cases} 0 & \text{for } K \text{ of type 1}, \\ 1 & \text{for } K \text{ of type 2}; \end{cases}$$

$$\alpha(p^n) = 0, \quad p \text{ is a prime}, \quad p \neq 3, \text{ and } \quad p \mid k.$$

If p is a prime and $p \equiv -1 \pmod 3$, $p \nmid k$,

$$\alpha(p^n) = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

If p is a prime and $p \equiv 1 \pmod 3$, $p \nmid k$,

$$\alpha(p^n) = 1 + n, \quad \text{when } (D|p)_3 = 1;$$

otherwise,

$$\alpha(p^n) = \begin{cases} 1, & n \equiv 0 \pmod 3, \\ -1, & n \equiv 1 \pmod 3, \\ 0, & n \equiv -1 \pmod 3. \end{cases}$$

Mention should also be made of a special divisibility property of h. In order to do this we first need some notation.

Let the number of distinct primes which divide k be ω, the number of distinct primes of the form $3t + 1$ which divide k be ω_1, and the number of distinct primes dividing k which are congruent to either ± 2 or ± 4 modulo 9 be ω_0. If $\omega_0 = 0$, put $\epsilon = 0$; otherwise, put $\epsilon = 1$; also, put $\omega^* = \omega - 1 - \epsilon$. Let r_n be the 3^n rank of the class group of K, and let $r = \Sigma r_n$; then $3^r \| h$.

It has been shown [3], [10], [11], [8] that

$$\max(\omega_1, \omega^*) \leqslant r_1 \leqslant \omega_1 + \omega^*. \tag{2.1}$$

Hence, if $\nu = \max(\omega_1, \omega^*)$, then $3^\nu | h$.

3. Estimation of $\Phi(1)$ for $Q(D^{1/3})$. We start by defining the multiplicative functions $\beta(j)$ and $\beta^*(j)$. We first put $\beta^*(1) = \beta(1) = 1$. If p is a prime and $p \equiv -1 \pmod 3$, we define

$$\beta^*(p^n) = \beta(p^n) = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

If p is a prime and $p \equiv 1 \pmod 3$, we define

$$\beta^*(p^n) = \beta(p^n) = 1 + n.$$

Finally, we define $\beta^*(3^n) = 0$ and $\beta(3^n) = 1$. If K is of type 1, we have $-\frac{1}{2}\beta^*(j) \leqslant \alpha(j) \leqslant \beta^*(j)$; and if K is of type 2, we have $-\frac{1}{2}\beta(j) \leqslant \alpha(j) \leqslant \beta(j)$.

We also define $M(m) = \max((\beta(t)/t), t \geqslant m)$. It can be shown that

$$M(m) = \max((\beta(t)/t), \quad t = m, m + 1, m + 2, \ldots, 3m).$$

This permits us to calculate $M(m)$ easily. It is evident that $|\alpha(j)/j| \leqslant M(m)$ for $j \geqslant m$.

Now $\Phi(s)$, the so-called Artin L function, is an entire function (Dedekind) and satisfies the functional equation

$$\Phi(1 - s) = C^{-2s+1}(\Gamma(s)/\Gamma(1 - s))\Phi(s);$$

and from a result of Barrucand [2], it follows that

$$\Phi(1) = \sum_{j=1}^{\infty} \frac{\alpha(j)}{j} e^{-jC} + C \sum_{j=1}^{\infty} \alpha(j)E(jC),$$

where $E(y) = \int_y^\infty (e^{-x}/x)\,dx$. Putting

$$A(n) = \sum_{j=1}^{n} \frac{\alpha(j)}{j} e^{-jC} + C \sum_{j=1}^{n} \alpha(j)E(jC),$$

we have

$$|\Phi(1) - A(n)| \leqslant S_1 + CS_2 + T_1 + CT_2,$$

where

$$S_1 = \sum_{j=n+1}^{m-1} \left|\frac{\alpha(j)}{j}\right| e^{-jC}, \qquad S_2 = \sum_{j=n+1}^{m-1} |\alpha(j)|E(jC),$$

$$T_1 = \sum_{j=m}^{\infty} \left|\frac{\alpha(j)}{j}\right| e^{-jC}, \qquad T_2 = \sum_{j=m}^{\infty} |\alpha(j)|E(jC).$$

Since $e^{-y}/y > E(y)$ for $y > 0$, it follows that

$$T_1 + CT_2 < 2M(m) \sum_{j=m}^{\infty} e^{-jC} = \frac{2M(m)e^{-(m-1)C}}{e^C - 1}.$$

If we put

$$H(m) = \frac{2M(m)}{C} e^{-(m-1)C} \Big/ (e^C - 1),$$

$$G(n, m, x) = \frac{\sqrt{3}x}{2\pi} \sum_{j=n+1}^{m-1} \beta(j)e^{-(2\pi j/\sqrt{3}x)} + \sum_{j=n+1}^{m-1} \beta(j)E(2\pi j/\sqrt{3}x),$$

$$G^*(n, m, x) = \frac{3\sqrt{3}x}{2\pi} \sum_{j=n+1}^{m-1} \beta^*(j)e^{-(2\pi j/3\sqrt{3}x)} + \sum_{j=n+1}^{m-1} \beta^*(j)E(2\pi j/3\sqrt{3}x),$$

we see that

$$\left|\frac{\Phi(1)}{C} - \frac{A(n)}{C}\right| < \begin{cases} G^*(n, m, x) + H(m) & \text{if } K \text{ is of type 1,} \\ G(n, m, x) + H(m) & \text{if } K \text{ is of type 2.} \end{cases}$$

If n is selected such that

$$|\Phi(1)/C - A(n)/C| < \tfrac{1}{2}3^{\nu}R,$$

then h is the unique integer in the interval $(h^* - \frac{1}{2}3^{\nu}, h^* + \frac{1}{2}3^{\nu})$ such that $3^{\nu}|h$. Here $h^* = A(n)/RC$.

4. Tabulation of h for $D = 2, 3, \ldots, 9999$. Values of $M(m)$ were calculated for $m = 1, 2, \ldots, 10^5$ and it was seen that for $D < 10^4$ and $m = \min([25/C], 10^5)$, the value of $H(m)$ was usually less than .1. The functions $G(n, m, x)$ and $G^*(n, m, x)$ were also tabulated for $x = 500(500)10^4$ and $n = x/10, 2x/10, 3x/10, \ldots, 2x$ by putting

$$m = \min\left(\left[\frac{25\sqrt{3}x}{2\pi}\right], 10^5\right) \quad \text{for } G,$$

and

$$m = \min\left(\left[\frac{75\sqrt{3}x}{2\pi}\right], 10^5\right) \quad \text{for } G^*.$$

All of these tables were calculated by computer and then stored on a disk file for easy access. The following computer algorithm was then used to determine h.

For any value of D, the computer first calculated the value of R by using Voronoi's Algorithm (see [4]). It then put x equal to the first multiple of 500 which exceeded ab and selected n from the stored tables of G, G^* and M to be the least value of $ix/10$ $(i = 1, 2, 3, \ldots, 20)$ such that

$$\tfrac{1}{2}3^{\nu}R > \begin{cases} G^*(n, m, x) + H(m) + .05 & \text{when } K \text{ is of type 1,} \\ G(n, m, x) + H(m) + .05 & \text{when } K \text{ is of type 2.} \end{cases}$$

With this value of n, $A(n)$ and subsequently h were easily computed.

The algorithm was implemented on an IBM/370-158 computer and run for all values of D such that $a > b$ between 1 and 10, 000. The programs were written in FORTRAN with special assembler language routines being used for evaluating $E(jC)$ and $\alpha(j)$; also, all calculations were performed in double precision. The calculation of all the regulators required eight hours of CPU time and four additional hours were required for the evaluation of the class numbers.

Some indication of the speed of this method is provided by noting that the technique described in [4] required 8 minutes to evaluate the regulators for $Q(D^{1/3})$ for $D = 2, 3, \ldots, 999$ and 42 additional minutes to evaluate the class numbers. Our current method required 8 minutes to obtain the regulators and only 5 additional minutes to obtain the class numbers for these values of D.

5. Evaluation of h by Using the Euler Product. Another formula for $\zeta_K(1)/\zeta(1)$, where $K = Q(D^{1/3})$, is given by the Euler product

$$\Phi(1) = \zeta_K(1)/\zeta(1) = \prod_q f(q),$$

where the product is taken over all the primes. Thus, for a given value of D, we define $f(q)$ for prime values of q by the following formulas:

$$\text{if } q \mid k, \quad f(q) = 1;$$
$$\text{if } 3 \nmid k, \quad f(3) = 3/2;$$
$$\text{if } q \equiv -1 \pmod 3, \quad f(q) = q^2/(q^2 - 1);$$
$$\text{if } q \equiv 1 \pmod 3, \quad f(q) = \begin{cases} q^2/(q-1)^2 & \text{when } (D|q)_3 = 1, \\ q^2/(q^2 + q + 1) & \text{when } (D|q)_3 \neq 1. \end{cases}$$

If we put the partial product $P(Q) = \Pi_q^Q f(q)$, we can estimate a value for $\Phi(1)$ by using a sufficiently large value of Q. The difficulty in using this formula lies in determining how big Q should be. In [15] Shanks made use of a similar Euler product to obtain class numbers for some special cubic fields. He (private communication) evaluated his partial Euler products using the first 500, 1000, 1500, etc., primes. When his estimate for the class number remained within .1 of the same integer for 6 consecutive partial Euler products, he took this integer as the class number.

We used this same criterion for estimating $\Phi(1)$ and discovered that in most cases only 3000 primes were needed in order to evaluate h. This method is very simple to program and executes from two to five times more rapidly than the technique described in Sections 3 and 4. It must, however, be emphasized that this procedure is not as rigorous as our first method. We cannot obtain bounds for the error on using the partial product that are as useful as the bounds we can obtain using our formulas for $A(n)$. The criterion which we do use for estimating $\Phi(1)$ by $P(Q)$ is one which is simply convenient for our calculations.

If, after Neild and Shanks [13], we put $E(Q) = 1000(P(Q) - \Phi(1))/\Phi(1)$, the relative error in parts per thousand on using $P(Q)$ to approximate $\Phi(1)$; and if we use 5000 primes to evaluate $E(Q)$, we get the following distribution for $E(Q)$ for the 166 pure cubic fields $Q(D^{1/3})$ where $8000 \leqslant D \leqslant 8200$.

$Q = 48611$

$E(Q)$

−3½ to −3	−3 to −2½	−2½ to −2	−2 to −1½	−1½ to −1	−1 to −½	−½ to 0	0 to ½	½ to 1	1 to 1½	1½ to 2	2 to 2½	2½ to 3	3 to 3½
1	0	3	8	19	23	31	23	17	16	19	5	0	1

This distribution is typical of the sort of distribution we get for $E(Q)$ for the pure cubic fields $Q(D^{1/3})$ $(2 \leqslant D \leqslant 10^4)$.

6. Some Results of the Calculations. A large table, giving for each D between 1 and 10^4 such that $a > b$, the value of k, the length of Voronoi's continued fraction period for $D^{1/3}$, the regulator of $Q(D^{1/3})$, the class number, and $\Phi(1)$, has been deposited in the U.M.T. file. We present here some selected results from that table. In Table 1 we give each value of h which occurs in the large table, the frequency $g(h)$ with which this h occurs, and the least value of D such that h is the class number for $Q(D^{1/3})$.

If in this table we had presented, instead of D, the least value of $|\Delta|$ (in the range of Δ being considered) such that h is the class number of the pure cubic field with discriminant Δ, we would find that in most cases this value of Δ would correspond to the discriminant of the cubic field $Q(D^{1/3})$ for the already given value of D. There are, however, several exceptions to this; for example, if $h = 4$, the least D value is 113, but the least $|\Delta|$ is $3 \cdot 233^2$ not $3 \cdot 339^2$.

TABLE 1

h	$g(h)$	D	h	$g(h)$	D
1	596	2	33	19	1618
2	285	11	34	1	1719
3	1847	7	36	262	322
4	87	113	37	2	5545
5	37	263	39	7	2597
6	952	39	40	2	2733
7	26	235	41	1	6659
8	32	141	42	21	515
9	1258	70	44	2	4817
10	9	303	45	68	763
11	7	2348	48	30	561
12	359	43	49	1	8171
13	5	1049	51	4	1037
14	7	514	52	1	4793
15	97	267	54	172	614
16	9	681	56	2	857
17	1	8511	57	5	1541
18	674	65	58	1	6814
19	2	667	60	14	997
20	6	761	63	29	1005
21	51	213	64	1	9749
22	4	281	66	5	3482
24	96	229	68	1	9521
26	1	3403	69	4	3590
27	385	182	70	1	3467
28	6	509	71	1	3539
30	38	524	72	90	741
32	3	2399	74	1	3581
			75	3	1657

TABLE 1 (*continued*)

h	$g(h)$	D	h	$g(h)$	D
78	9	1801	216	17	2765
80	1	4799	222	1	5823
81	77	1298	225	3	5362
84	9	1737	230	1	4451
87	2	4103	240	2	5835
90	27	970	243	6	3913
93	1	2748	252	3	2786
96	5	4307	254	1	8002
99	8	995	255	1	2751
102	4	2374	264	1	7297
105	4	2737	270	4	4593
108	87	511	276	1	4093
111	2	5737	279	1	5149
117	4	5215	288	1	5826
120	10	1727	297	3	6487
126	23	1141	300	1	9931
127	1	2741	306	2	4694
128	1	5987	312	1	9938
129	1	2946	315	2	5359
132	3	3045	324	10	2198
135	11	1015	336	1	8005
136	1	3209	342	1	3907
141	1	6991	351	3	3605
144	17	1730	360	1	7985
150	1	8431	369	1	5829
153	2	3661	372	1	7133
154	1	9041	378	3	3155
156	2	7461	390	1	9591
162	36	813	396	2	7997
168	2	2747	405	6	7970
171	1	9198	432	4	6878
175	1	5711	435	1	8006
180	12	2702	459	1	9254
186	1	4099	480	1	7415
189	7	6430	486	4	6162
192	2	7925	576	1	4291
198	7	3374	585	1	9262
201	2	2723	612	1	7995

TABLE 1 (*continued*)

h	$g(h)$	D	h	$g(h)$	D
630	1	9933	972	1	9709
648	1	4097	1017	1	8615
696	1	5503	1170	1	7999
747	1	2743	1296	1	8827
756	1	8030			

In Table 1A we give the frequency $v(p)$, $p \leqslant 19$, of the cases where $p|h$ and the percentage of the occurrences.

TABLE 1A

p	$v(p)$	%	p	$v(p)$	%
2	3510	43.22	11	62	0.76
3	6954	85.62	13	36	0.44
5	369	4.54	17	19	0.23
7	202	2.49	19	9	0.11

Moreover, we have 7409 fields such that $h = 2^\alpha 3^\beta$ and 713 fields such that some prime >3 divides h.

In Table 2 we give the number m of values of D in the ranges $1000(i - 1)$ to $1000i$ ($i = 1, 2, 3, \ldots, 10$) for which the class number of $Q(D^{1/3})$ is unity.

TABLE 2

Range of D	m	Range of D	m
0-1000	98	5000—6000	55
1000-2000	64	6000—7000	49
2000-3000	56	7000—8000	54
3000-4000	61	8000—9000	44
4000-5000	65	9000-10000	50

Denote by $R(d)$ and $\Phi_d(1)$ the value of the regulator and the value of $\Phi(1)$ for $Q(d^{1/3})$, respectively. In Table 3 we give those values of D and $\Phi_D(1)$ such that

$$\Phi_D(1) < \Phi_d(1) \quad \text{for all } 0 < d < D.$$

TABLE 3

D	$\Phi(1)$	D	$\Phi(1)$
2	0.8146240593	1510	0.6672743355
74	0.7323553491	2740	0.6496367445
166	0.6767319520	4630	0.6251252454
276	0.6733957020	9770	0.6199747135
830	0.6684533198		

In Table 4 we give those values of D and $\Phi_D(1)$ such that $\Phi_D(1) > \Phi_d(1)$ for all $0 < d < D$.

TABLE 4

D	$\Phi(1)$	D	$\Phi(1)$
2	0.8146240593	307	2.8227637445
3	1.0176145615	559	2.9367139608
5	1.1637304168	629	2.9626689819
6	1.1668639154	827	2.9707482692
7	1.2650247640	883	3.0623905474
13	1.5743940270	1009	3.0683245965
29	1.6791537873	1457	3.0931407438
35	1.7499243062	1513	3.3383074285
53	1.8171807443	1945	3.3680857678
55	2.1254129939	3457	3.5990411752
71	2.2034566301	4789	3.6257791705
127	2.3311172521	5669	3.7254983552
181	2.6622437425	9017	3.8119134914

Finally, in Table 5 we give those values of D, $R(D)$ and J, the length of Voronoi's algorithm period of $D^{1/3}$, such that

$$R(D) > R(d) \quad \text{for all } 0 < d < D.$$

TABLE 5

D	$R(D)$	J	D	$R(D)$	J
2	1.347377348	1	951	1521.5849715	1352
3	2.524681405	3	1163	1818.3574652	1595
5	4.811986540	5	1301	2549.9434350	2307
6	5.789932142	5	1721	3669.3791260	3320
15	9.692951678	5	2003	3675.2829265	3255
23	22.595071214	21	2283	4340.6136141	3959
29	40.270821121	35	2927	4671.7189737	4076
41	56.289370200	49	3543	4681.9661909	4096
69	103.810793808	100	3557	6170.2103314	5393
137	134.626355970	122	3821	7106.2863230	6388
167	220.571825346	206	3921	8909.1586123	8014
227	224.944023983	206	4523	9440.9625040	8545
239	431.942240996	390	5153	9766.3676264	8576
411	555.643020852	488	5433	12019.3087665	10702
419	711.993772506	646	6999	13777.0095919	12338
447	778.588027713	719	8093	15231.6425197	13591
573	991.930184538	877	8429	17248.5337519	15481
771	1321.452703846	1202			

7. Some Results on the Class Structure. We shall employ in this section the same notation as that introduced in Section 2. We also define $j = e^{2\pi i/3}$ and let the symbols p, q represent primes. By $(n|l)_3$ we represent a cubic character modulo l, where l is either a prime of the form $1 + 3t$ or $l = 9$. Since $(n|l)_3 = j, j^2$, or 1, the character is completely determined when we select a cubic nonresidue n_0 and define the value of $(n_0|l)$ as either j or j^2.

From the inequality (2.1), we see that r_1 is known exactly in four cases.

(1) $\omega_1 = 0$, $r_1 = \omega^*$.

(2) $D = p$, $p \equiv 1 \pmod 9$, $r_1 = 1$.

(3) $D = p, 3p, 9p$, $p \equiv 4, 7 \pmod 9$, $r_1 = 1$.

(4) $D = pp', pp'^2$, $D \equiv 1 \pmod 9$, $\epsilon = 1$, $r_1 = 2$.

In the other cases, it was shown by Gerth [8] and Kobayashi [11] that $r_1 = \omega_1 + \omega^* - s$, where s can be computed by evaluating Hilbert symbols in $Q(\sqrt{-3})$ or, what is the same thing, cubic residuacity symbols in $Q(\sqrt{-3})$. The case of $\epsilon = 0$ is troublesome; consequently, we shall restrict ourselves to the case of $\epsilon = 1$ only. If $\omega_1 = 1$, we have $k = pk^*$, where p is a prime congruent to 1 modulo 3 and no prime which divides k^* is congruent to 1 modulo 3. From this we can deduce

THEOREM 1. *If $D = p, 3p, 9p$, where $p \equiv 4, 7 \pmod 9$ and $(3|p)_3 \neq 1$, then $r_1 = 1, r_2 = 0$ and, consequently, $3 \| h$.*

THEOREM 2. *If $D = pq$, where $p \equiv -q \equiv 1 \pmod 3$, $(q|p)_3 \neq 1$, and $\epsilon = 1$, then $r_1 = 1, r_2 = 0$.*

THEOREM 3. *If $D = pq \not\equiv \pm 1 \pmod 9$, where $q \equiv 2 \pmod 3$, $p \equiv 4, 7 \pmod 9$, $(3|p)_3 = 1$ and $(q|p)_3 \neq 1$, the ideal ramifying (3) in $K = Q(D^{1/3})$ is principal.*

To prove Theorem 1, we use the fact that $r_1 = 1$ and remark that we have a rational genus character system, that is if $\mathfrak{p}$ is a principal ideal in $K = Q(D^{1/3})$, then $(N(\mathfrak{p})|p)_3 = 1$. Since $(3|p)_3 \neq 1$, we have an "ambiguous" class C1 such that if $\mathfrak{p}^* \in$ C1, then $(N(\mathfrak{p}^*)|p)_3 = j$—a contradiction. (The proof may be compared to the well-known proof of the result that $2 \| h(\sqrt{-p})$ when $p \equiv 5 \pmod 8$.)

TABLE 6

p,q	$(p\|9)_3$	$(q\|9)_3$	$(3\|p)_3$	$(3q\|p)_3$	h
7,2	j	j	j	1	3
13,2	j	j^2	j	j^2	3
7,11	j	j	j	j	3
61,2	j	j	1	j	12
67,2	j	j^2	1	j	3
73,2	1	j	1	j	3
19,2	1	j	j	j^2	3
19,23	1	j	j	1	6
7,17	j	1	j	j^2	3
13,17	j	1	j	1	3
61,17	j	1	1	j	51

To prove Theorem 2, we require a list of D values in which all possibilities for the distribution of characters are present. It can be verified in Table 6 that, if $(n|p)_3$ is defined by $(q|p)_3 = j$, in each case we have $3\|h$. The proof that $r_2 = 0$ is similar to that of Theorem 1. Theorem 3 is a simple corollary of Theorem 2.

8. Remarks and Conjectures. We present in this section some conjectures based on observations made of various phenomena in the table and verified for all $D \leqslant 9999$.

Conjecture 1. If $D = p, 3p, 9p$, where $p \equiv 4, 7 \pmod 9$ and $(3|p)_3 = 1$, then $r_2 = 0$.

Conjecture 2. If $D = pp'$, where $p \equiv p' \equiv 1 \pmod 3$ and $D \not\equiv 1 \pmod 9$, then $27|h$ if and only if $(p|p')_3 = (p'|p)_3$ or if $p \equiv 1 \pmod 9$, $(3|p)_3 = (p'|p)_3 = 1$.

Both Theorem 1 and Conjecture 1 are false if $p \equiv 1 \pmod 9$; for, in some cases we have $9|h$ which implies that $r_2 \geqslant 1$. This happens for $p = 199, 271, \ldots$. Similarly, Conjecture 2 becomes false when $pp' \equiv 1 \pmod 9$ even if $\epsilon = 1$, as may be inferred from the fact that $h = 27$ for $D = 7.31$.

With the exception of the results of Theorems 1 and 2, almost nothing appears to be known about r_2; however, it is perhaps worth mentioning that $r_2 = 1$ for some $D = qq'q''$, where $q \equiv q' \equiv q'' \equiv -1 \pmod 3$. For example if $D = 2 \cdot 5 \cdot 101$, $h = 54$ and $r_1 = 2$.

Conjecture 3. If $p \equiv 1 \pmod 9$ and $q \equiv -1 \pmod 9$, $(q|p)_3 \neq 1, D = pq$, then $r_1 = 1$ and $r_2 = 0$.

Conjecture 4. If $q \equiv q' \equiv 2$ or $5 \pmod 9$ then $r_1 = 1, r_2 = 0$.

A part of these conjectures may be proved by using some unpublished results of Gerth [9], but no theory concerning r_2 seems to be known. For example, if $D = p \equiv 1 \pmod 9$, we may find $r_2 = 0$ ($p = 19$) or $r_2 = 1$ ($p = 199$).

Generally, the class number remains small; but in some rare cases it may be unusually large as in the case of $D = 8827, h = 1296$.

9. Acknowledgment. The authors would like to thank Daniel Shanks for making several suggestions concerning this work and for pointing out some errors which occurred in our early computer runs.

151 Rue Chateau des Rentiers
Paris 75013, France

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

1. I. O. ANGELL, "A table of complex cubic fields," *Bull. London Math. Soc.*, v. 5, 1973, pp. 37–38. MR **47** #6648.

2. PIERRE BARRUCAND, "Sur certaines séries de Dirichlet," *C. R. Acad. Sci. Paris Sér. A–B*, v. 269, 1969, pp. A294–A296. MR **40** #101.

3. P. BARRUCAND & H. COHN, "A rational genus, class number divisibility, and unit theory for pure cubic fields," *J. Number Theory*, v. 2, 1970, pp. 7–21. MR **40** #2643.

4. B. D. BEACH, H. C. WILLIAMS & C. R. ZARNKE, "Some computer results on units in quadratic and cubic fields," *Proc. Twenty-Fifth Summer Meeting of the Canadian Math. Congress* (Lakehead Univ., Thunder Bay, Ont., 1971), Lakehead Univ., Thunder Bay, Ont., 1971, pp. 609–648. MR **49** #2656.

5. J. W. S. CASSELS, "The rational solutions of the diophantine equation $Y^2 = X^3 - D$," *Acta Math.*, v. 82, 1950, pp. 243–273. MR **12**, 11.

6. HARVEY COHN, "A numerical study of Dedekind's cubic class number formula," *J. Res. Nat. Bur. Standards*, v. 59, 1957, pp. 265–271. MR **19**, 944.

7. R. DEDEKIND, "Über die Anzahl der Idealklassen in reinen kubischen Zahlkörpern," *J. Reine Angew. Math.*, v. 121, 1900, pp. 40–123.

8. FRANK GERTH III, "Ranks of Sylow 3 subgroups of ideal class groups of certain cubic fields," *Bull. Amer. Math. Soc.*, v. 79, 1973, pp. 521–525.

9. FRANK GERTH III, "The class structure of the pure cubic field." (Unpublished.)

10. TAIRA HONDA, "Pure cubic fields whose class numbers are multiples of three," *J. Number Theory*, v. 3, 1971, pp. 7–12. MR **45** #1877.

11. S. KOBAYASHI, "On the 3-rank of the ideal class groups of certain pure cubic fields," *J. Fac. Sci. Univ. Tokyo Sect. I A Math.*, v. 20, 1973, pp. 209–216. MR **48** #3919.

12. A. MARKOV, "Sur les nombres entiers dépendants d'une racine cubique d'un nombre entier ordinaire," *Mem. Acad. Imp. Sci. St. Petersburg*, v. (7) 38, 1892, no. 9, pp. 1–37.

13. CAROL NEILD & DANIEL SHANKS, "On the 3-rank of quadratic fields and the Euler product," *Math. Comp.*, v. 28, 1974, pp. 279–291.

14. ERNST S. SELMER, "Tables for the purely cubic field $K(\sqrt[3]{m})$," *Avh. Norske Vid. Akad. Oslo I*, v. 1955, no. 5. MR **18**, 286.

15. DANIEL SHANKS, "The simplest cubic fields," *Math. Comp.*, v. 28, 1974, pp. 1137–1152.