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Converting Interpolation Series into Chebyshev Series by Recurrence Formulas

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Abstract. Interpolation series (divided difference, Gregory-Newton, Gauss, Stirling, Bessel) are converted into Chebyshev (or Jacobi) series by applying a previously derived general five-term recurrence formula [3]. It employs the coefficients in three-term linear recurrence formulas (same kind as for orthogonal polynomials) which have been found for the mth degree nonorthogonal polynomial coefficients of the differences used in the interpolation series. In the Gauss, Stirling and Bessel series, the coefficients in the recurrence formulas vary with the parity of m. The basic five-term recurrence formula is applicable also to: (1) inter- and intraconversion of power series in $ax + b$, divided difference and equal-interval interpolation series (including subtabulation), and Chebyshev series, (2) obtaining Chebyshev series for solutions of difference equations, (3) the derivation of formulas for Chebyshev coefficients in terms of differences, and (4) the conversion of interpolation series into Chebyshev series, for more than one variable.

Introduction. In a previous note on converting from one orthogonal polynomial series $\Sigma_{m=0}^{n} a_m q_m(x)$ into another orthogonal polynomial series $\Sigma_{m=0}^{n} A_m Q_m(x)$ [3], the basic five-term recurrence formula [3, (22)] is deduced just from these three-term recurrence formulas for $q_m \equiv q_m(x)$ and $Q_m \equiv Q_m(x)$:

$$\text{(1a)} \qquad q_{-1} = 0, \quad q_{m+1} + (a(m) + b(m)x)q_m + c(m)q_{m-1} = 0,$$

$$\text{(1b)} \quad Q_{-1} = 0, \quad Q_{m+1} + (A(m) + B(m)x)Q_m + C(m)Q_{m-1} = 0, \quad m = 0(1)n - 1.$$

Since the orthogonality of q_m or Q_m is not necessary for (1a) or (1b) respectively, it happens that [3, (22)] has many applications when either of, or both, q_m and Q_m are not orthogonal. We give here some useful applications to converting a number of different interpolation series into Chebyshev series. These seven interpolation series, namely, Newton's divided difference formula, the Gregory-Newton formulas with forward and backward differences, Gauss's forward and backward formulas, Stirling's formula and Bessel's formula [2], through the nth degree terms,[1] are each expressible as $\Sigma_{m=0}^{n} a_m q_m$ where q_m satisfies a three-term recurrence relation of the form (1a). This

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[1] Remainder terms are not considered here.

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is not so obvious in the Gauss, Stirling and Bessel formulas, especially since $a(m)$, $b(m)$ and $c(m)$ in (1a) depend also upon the parity of m.

Notation. We retain here the same notations as in [3], except for the variable p in place of the variable x, where $p = (x - x_0)/h$ for the tabular interval h, in all the equal-interval interpolation formulas (i.e., in all but the divided difference formula). We express all interpolation formulas in a notation that is slightly different from that in [2] and also somewhat more compact, but which expresses the q_m in $P_n(x)$ or $P_n(x_0 + ph) = \Sigma_{m=0}^{n} a_m q_m$ directly in terms of m and x or m and p, respectively. All differences or divided differences, as well as Chebyshev or other coefficients, are those of the nth degree $(n + 1)$-point interpolation polynomial $P_n(x)$ which is not the same as $f(x)$, the function that is tabulated, when $f(x)$ is not an exact nth degree polynomial.[2] Thus, here $\mathcal{D}_x^m$ denotes the mth divided difference of $P_n(x)$ for $(x_0, x_1, \ldots, x_m)$. Other difference notations are standard, such as Δ_0^m or ∇_0^m for the mth forward or backward difference of $P_n(x_0 + ph)$ at $x = x_0$, and δ_0^m or $\mu\delta_0^m$ and $\delta_{1/2}^m$ or $\mu\delta_{1/2}^m$ for mth central or mean central differences of $P_n(x_0 + ph)$ at x_0 and $x_0 + \frac{1}{2}h$, respectively.

Recurrence Scheme for Conversion of Series. To make this note self-sufficient, the previously published general five-term recurrence scheme [3, (22)], for obtaining $A_m = a_m^{(n)}$ from a_m, $m = 0(1)n$, is given here in this more condensed form:

$$
\begin{aligned}
a_m^{(k+1)} = -a_m^{(k-1)}c(n-k) &+ a_{m-1}^{(k)}b(n-k-1)/B(m-1) \\
&+ a_m^{(k)}[-a(n-k-1) + b(n-k-1)A(m)/B(m)] \\
&+ a_{m+1}^{(k)}b(n-k-1)C(m+1)/B(m+1) + \delta_m^0 a_{n-k-1}, \\
&m = 0(1)k+1, \qquad k = -1(1)n-1,
\end{aligned} \tag{1c}
$$

where $a_i^{(j)} = 0$ for $i < 0$ or $i > j$, and $\delta_m^0 = 0$, $m \neq 0$, $\delta_m^0 = 1$, $m = 0$.

Interpolation Polynomials and Recurrence Coefficients. Following are the seven above-mentioned interpolation polynomials, each written in the form $\Sigma_{m=0}^{n} a_m q_m(x \text{ or } p)$,[3] together with the corresponding coefficients $a(m)$, $b(m)$ and $c(m)$ in the recurrence formula (1a). In every case $a_0 = P_n(x_0) = f(x_0) = \Delta_0^0 = \nabla_0^0 = \delta_0^0$, $q_0 = 1$.

I. *Newton Divided Difference Formula.*

$$
P_n(x) = P_n(x_0) + \sum_{m=1}^{n} \left[\prod_{i=0}^{m-1} (x - x_i) \right] \mathcal{D}_x^m, \tag{2}
$$

$$
a_m = \mathcal{D}_x^m, \quad q_m = \prod_{i=0}^{m-1} (x - x_i), \qquad m = 1(1)n, \tag{3}
$$

[2] Even though initially $P_n(x)$ must have the identical n differences of $f(x)$ upon which $P_n(x)$ is constructed, the point here being that all subsequent series conversions are for $P_n(x)$, and not $f(x)$.

[3] Strictly speaking, the form is $\Sigma_{m=0}^{n} q_m(x \text{ or } p) a_m$ since, by custom, the differences a_m are written after the mth degree polynomial coefficients $q_m(x \text{ or } p)$.

(4) $$a(m) = x_m, \quad b(m) = -1, \quad c(m) = 0, \quad m = 0(1)n - 1.$$

IIa. *Newton Forward Difference Formula.*

(5) $$P_n(x_0 + ph) = P_n(x_0) + \sum_{m=1}^{n} \binom{p}{m} \Delta_0^m,$$

(6) $$a_m = \Delta_0^m, \quad q_m = \binom{p}{m}, \quad m = 0(1)n,$$

(7) $$a(m) = m/(m + 1), \quad b(m) = -1/(m + 1), \quad c(m) = 0, \quad m = 0(1)n - 1.$$

IIb. *Newton Backward Difference Formula.*

(8) $$P_n(x_0 + ph) = P_n(x_0) + \sum_{m=1}^{n} \binom{p + m - 1}{m} \nabla_0^m,$$

(9) $$a_m = \nabla_0^m, \quad q_m = \binom{p + m - 1}{m}, \quad m = 0(1)n,$$

(10) $$a(m) = -m/(m + 1), \quad b(m) = -1/(m + 1), \quad c(m) = 0, \quad m = 0(1)n - 1.$$

IIIa. .*Gauss Forward Formula.*

(11) $$P_n(x_0 + ph) = P_n(x_0) + \binom{p}{1}\delta_{1/2} + \binom{p}{2}\delta_0^2 + \binom{p + 1}{3}\delta_{1/2}^3 + \binom{p + 1}{4}\delta_0^4 + \cdots,$$

(12a) $$a_m = \delta_0^m, \quad q_m = \binom{p + m/2 - 1}{m}, \quad m \text{ even},$$[4]

(12b) $$a_m = \delta_{1/2}^m, \quad q_m = \binom{p + (m - 1)/2}{m}, \quad m \text{ odd}.$$

(13a) $$a(m) = -m/2(m + 1), \quad b(m) = -1/(m + 1), \quad c(m) = 0, \quad m \text{ even},$$

(13b) $$a(m) = 1/2, \quad b(m) = -1/(m + 1), \quad c(m) = 0, \quad m \text{ odd}.$$

IIIb. *Gauss Backward Formula.*

(14) $$P_n(x_0 + ph) = P_n(x_0) + \binom{p}{1}\delta_{-1/2} + \binom{p + 1}{2}\delta_0^2 + \binom{p + 1}{3}\delta_{-1/2}^3 + \binom{p + 2}{4}\delta_0^4 + \cdots,$$

(15a) $$a_m = \delta_0^m, \quad q_m = \binom{p + m/2}{m}, \quad m \text{ even},$$

(15b) $$a_m = \delta_{-1/2}^m, \quad q_m = \binom{p + (m - 1)/2}{m}, \quad m \text{ odd}.$$

(16a) $$a(m) = m/2(m + 1), \quad b(m) = -1/(m + 1), \quad c(m) = 0, \quad m \text{ even},$$

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[4] Includes $m = 0$ except where > 0 is indicated.

(16b) $a(m) = -1/2, \quad b(m) = -1/(m+1), \quad c(m) = 0, \quad m$ odd.

IV. *Stirling Formula.*

(17)
$$P_n(x_0 + ph) = P_n(x_0) + \binom{p}{1}\mu\delta_0 + \frac{p}{2}\binom{p}{1}\delta_0^2 + \binom{p+1}{3}\mu\delta_0^3 + \frac{p}{4}\binom{p+1}{3}\delta_0^4 + \cdots,$$

(18a) $$a_m = \delta_0^m, \quad q_m = \frac{p}{m}\binom{p+m/2-1}{m-1}, \quad m \text{ even } (> 0),$$

(18b) $$a_m = \mu\delta_0^m, \quad q_m = \binom{p+(m-1)/2}{m}, \quad m \text{ odd}.$$

(19a) $a(m) = 0, \quad b(m) = -1/(m+1), \quad c(m) = m/4(m+1), \quad m$ even,

(19b) $a(m) = 0, \quad b(m) = -1/(m+1), \quad c(m) = 0, \quad m$ odd.

V. *Bessel Formula.*

(20)
$$P_n(x_0 + ph) = P_n(x_0) + \binom{p}{1}\delta_{1/2} + \binom{p}{2}\mu\delta_{1/2}^2 + \frac{p(p-1/2)(p-1)}{3!}\delta_{1/2}^3 + \binom{p+1}{4}\mu\delta_{1/2}^4 + \cdots,$$

(21a) $$a_m = \mu\delta_{1/2}^m, \quad q_m = \binom{p+m/2-1}{m}, \quad m \text{ even } (> 0),$$

(21b′) $$a_1 = \delta_{1/2}, \quad q_1 = p,$$

(21b) $$a_m = \delta_{1/2}^m, \quad q_m = \frac{p-1/2}{m}\binom{p+(m-3)/2}{m-1}, \quad m \text{ odd } (> 1).$$

(22a′) $a(0) = 0, \quad b(0) = -1, \quad c(0) = 0,$

(22a) $a(m) = 1/2(m+1), \quad b(m) = -1/(m+1), \quad c(m) = 0, \quad m$ even (> 0).

(22b′) $a(1) = 1/2, \quad b(1) = -1/2, \quad c(1) = 0,$

(22b) $a(m) = 1/2(m+1), \quad b(m) = -1/(m+1), \quad c(m) = m/4(m+1), \quad m$ odd (> 1).

Choice of Interval for Chebyshev Polynomials. To convert the preceding interpolation polynomials into Chebyshev series by (1c), we need $A(m)$, $B(m)$ and $C(m)$ for the Chebyshev polynomials that have been obtained from $T_m \equiv T_m(x) = \cos(m \arccos x)$, $m \geqslant 0$, after the $[-1, 1]$ interval for x (or p) has been transformed linearly into an interval $[a, b]$ that is best suited to the interpolation formula. For T_m itself, when the best interval is $[-1, 1]$, we have the recurrence relation

(23) $$T_0 = 1, \quad T_1 = x, \quad T_{m+1} = 2xT_m - T_{m-1}, \quad m > 0,$$

so that

(24) $$A(0) = C(0) = 0, \quad B(0) = -1; \quad A(m) = 0, \quad B(m) = -2, \quad C(m) = 1, \quad m > 0.$$

For the interval $[a, b]$, T_m is transformed into $T_m^{(a,b)} \equiv T_m((2x - a - b)/(b - a))$, which satisfies the recurrence relation

(25) $$T_0^{(a,b)} = 1, \quad T_1^{(a,b)} = \frac{2x - a - b}{b - a},$$
$$T_{m+1}^{(a,b)} = \frac{4x - 2a - 2b}{b - a} T_m^{(a,b)} - T_{m-1}^{(a,b)}, \qquad m > 0,$$

so that

(26) $$A(0) = (b + a)/(b - a), \quad B(0) = -2/(b - a), \quad C(0) = 0;$$
$$A(m) = \frac{2(b + a)}{b - a}, \quad B(m) = -\frac{4}{b - a}, \quad C(m) = 1, \quad m > 0.$$

For I, it is natural to employ $T_m^{(a,b)}$ where $[a, b]$ is the smallest interval that includes x_i, $i = 0(1)n$, and x. For IIa and IIb, where p usually lies within $[0, 1]$, we convert to a series in $T_m^{(0,1)}$, often written as T_m^*, for which

(27) $$A(0) = 1, \quad B(0) = -2, \quad C(0) = 0;$$
$$A(m) = 2, \quad B(m) = -4, \quad C(m) = 1, \quad m > 0.$$

However, when IIa or IIb is used for both interpolation and extrapolation as far as one tabular interval, in which case p might be anywhere within the interval $[-1, 1]$, we convert to a series in $T_m^{(-1,1)} \equiv T_m$. For IIIa, p is most likely to be in $[0, 1/2]$ and the Chebyshev series is in terms of $T_m^{(0,1/2)}$, for which

(28) $$A(0) = 1, \quad B(0) = -4, \quad C(0) = 0;$$
$$A(m) = 2, \quad B(m) = -8, \quad C(m) = 1, \quad m > 0.$$

Similarly for IIIb, where p is within $[-1/2, 0]$, we transform to $T_m^{(-1/2,0)}$ for which

(29) $$A(0) = -1, \quad B(0) = -4, \quad C(0) = 0;$$
$$A(m) = -2, \quad B(m) = -8, \quad C(m) = 1, \quad m > 0.$$

For IV, p is most likely to lie within $[-1/2, 1/2]$, and we employ $T_m^{(-1/2,1/2)}$ for which

(30) $$A(0) = C(0) = 0, \quad B(0) = -2;$$
$$A(m) = 0, \quad B(m) = -4, \quad C(m) = 1, \quad m > 0.$$

But should p be confined to either of the intervals $[0, 1/2]$ or $[-1/2, 0]$, we employ, of course, $T_m^{(0,1/2)}$ or $T_m^{(-1/2,0)}$, respectively. For V, p is most likely to be within $[0, 1]$ for which we transform to $T_m^{(0,1)} \equiv T_m^*$. But if p should happen to be confined to either of the intervals $[0, 1/2]$ or $[1/2, 1]$, we should transform to $T_m^{(0,1/2)}$ or $T_m^{(1/2,1)}$, respectively, where for $T_m^{(1/2,1)}$,

$$(31)\qquad \begin{aligned} &A(0) = 3, \quad B(0) = -4, \quad C(0) = 0;\\ &A(m) = 6, \quad B(m) = -8, \quad C(m) = 1, \quad m > 0. \end{aligned}$$

For some purposes one might wish to convert the interpolation polynomial $P_n(x)$ into a series in other than Chebyshev polynomials, such as Jacobi polynomials $P_m^{(\alpha,\beta)} \equiv P_m^{(\alpha,\beta)}(x)$, for suitable α and β. For the interval $[-1, 1]$ we have

$$(32)\qquad \begin{aligned} &A(0) = -\tfrac{1}{2}(\alpha - \beta), \quad B(0) = -\tfrac{1}{2}(\alpha + \beta + 2), \quad C(0) = 0;\\ &A(m) = -(\alpha^2 - \beta^2)\,\frac{(2m + \alpha + \beta + 1)}{D(m)},\\ &B(m) = -(2m + \alpha + \beta)(2m + \alpha + \beta + 1)\,\frac{(2m + \alpha + \beta + 2)}{D(m)},\\ &C(m) = 2(m + \alpha)(m + \beta)\,\frac{(2m + \alpha + \beta + 2)}{D(m)},\\ &D(m) = 2(m + 1)(m + \alpha + \beta + 1)\,(2m + \alpha + \beta), \quad m > 0. \end{aligned}$$

As in the Chebyshev series above, to obtain a Jacobi series for an interval $[a, b]$ that is specially suited to the interpolation series, we employ $P_m^{(\alpha,\beta)}((2x(\text{or } 2p) - a - b)/(b - a))$, and for $m \geqslant 0$, $A(m)$ is replaced by $A(m) - (a + b)B(m)/(b - a)$, $B(m)$ by $2B(m)/(b - a)$, while $C(m)$ is unchanged. For a Jacobi series giving a good unweighted least-square type of approximation, we might choose $\alpha = \beta = 0$, i.e., the Legendre polynomials $P_m \equiv P_m^{(0,0)}(x)$, where for the interval $[-1, 1]$ we have

$$(33)\qquad \begin{aligned} &A(0) = C(0) = 0, \quad B(0) = -1;\\ &A(m) = 0, \quad B(m) = -\frac{(2m + 1)}{(m + 1)}, \quad C(m) = \frac{m}{(m + 1)}, \quad m > 0, \end{aligned}$$

and after transformation to $[a, b]$,

$$(34)\qquad \begin{aligned} &A(0) = \frac{b + a}{b - a}, \quad B(0) = -\frac{2}{b - a}, \quad C(0) = 0;\\ &A(m) = \frac{(2m + 1)(b + a)}{(m + 1)(b - a)}, \quad B(m) = -\frac{2(2m + 1)}{(m + 1)(b - a)},\\ &C(m) = \frac{m}{m + 1}, \quad m > 0. \end{aligned}$$

Other Related Applications of (1c).[5] It is worth noting that (1c) is also useful for interconversion of power series $\Sigma_{m=0}^{n} a_m(a + bx)^m$, divided difference series and Chebyshev series, including cases of intraconversion, namely, $\Sigma_{m=0}^{n} a_m(a + bx)^m$ into $\Sigma_{m=0}^{n} A_m(A + Bx)^m$,

$$P_n(x_0) + \sum_{m=1}^{n} \left[\prod_{i=0}^{m-1} (x - x_i)\right] \mathcal{D}_x^m$$

into

[5] Although, strictly speaking, this present section is not covered by the title of the article, its content is closely related, and it should be included here for a fuller picture of the value of (1c) for nonorthogonal series.

$$P_n(x_0') + \sum_{m=1}^{n} \left[\prod_{i=0}^{m-1} (x - x_i')\right] \mathcal{D}_{x'}^m,$$

and $\Sigma_{m=0}^n a_m T_m^{(a,b)}$ into $\Sigma_{m=0}^n A_m T_m^{(a',b')}$. Furthermore, (1c) is applicable to these three frequently occurring operations:

1) *Given the divided differences* $\mathcal{D}_x^m$, *to find the advancing or central differences for a prescribed tabular interval* h.[6] We find the divided difference series in terms of the variable p by replacing $\mathcal{D}_x^m$ by $\mathcal{D}_p^m = h^m \mathcal{D}_x^m$, x by p, and x_i by $p_i = (x_i - x_0)/h$ before applying (1c).[7]

2) *Conversion of an equal-interval interpolation series into a divided difference series.*[8] We first convert the former into a divided difference series in the variable p, based upon p_i in 1), namely, $P_n(x_0) + \Sigma_{m=1}^n [\Pi_{i=0}^{m-1} (p - p_i)] \mathcal{D}_p^m$, from which we obtain $\mathcal{D}_x^m = \mathcal{D}_p^m / h^m$.

3) *Obtaining the new differences after changing the tabular interval* h *to* h_1 *in any equal-interval interpolation formula (particularly, in subtabulation to a smaller interval).*[8] Writing the same interpolation polynomial for different intervals as

$$P_n(x_0 + ph) = \sum_{m=0}^{n} a_m q_m(p) \quad \text{and} \quad P_n(x_0 + Ph_1) = \sum_{m=0}^{n} A_m q_m(P),$$

from $P = (h/h_1)p$ and

$$(35) \quad q_{m+1}(P) + [a(m) + b(m)P]\, q_m(P) + c(m) q_{m-1}(P) = 0, \qquad m = 0(1)n - 1,$$

noting that $q_m(P)$ is also $Q_m(p)$ in (1b), we find that

$$(36) \quad A(m) = a(m), \qquad B(m) = (h/h_1) b(m) \quad \text{and} \quad C(m) = c(m), \qquad m = 0(1)n - 1.$$

Similar reasoning applies in the slightly more involved case where there is both change of interval and conversion to a formula that does not have the same kind of differences (e.g., from Gregory-Newton at interval h to Stirling at interval h_1).[8] We then have $P_n(x_0 + Ph_1) = \Sigma_{m=0}^n A_m Q_m(P) = \Sigma_{m=0}^n A_m \overline{Q}_m(p)$, and from

$$(37) \quad Q_{m+1}(P) + (A(m) + B(m)P) Q_m(P) + C(m) Q_{m-1}(P) = 0, \qquad m = 0(1)n - 1,$$

or

$$(38)\ \overline{Q}_{m+1}(p) + (A(m) + (h/h_1) B(m) p) \overline{Q}_m(p) + C(m) \overline{Q}_{m-1}(p) = 0, \quad m = 0(1)n - 1,$$

it is clear that we must replace only $B(m)$ by $(h/h_1)B(m)$ in (1c).[8]

For another and somewhat different application of (1c), we note that it may be useful in deriving Chebyshev series approximations to solutions of difference equations, on the assumption of a polynomial approximation to the solution, whose differences of

[6] The discussion here is restricted to the more usual case where x_0 is the same for both series.

[7] The use of (1c) might not be always recommended for 1) and 3), save for checking or where it happens to be a more stable method. The reason is that it appears easier, in some cases, to compute $P_n(x)$ at the newly located equally-spaced points x_i and then to obtain the differences, there being no divisions for 1) and divisions by small factorials for 3).

[8] (See footnote 6 above.)

some special kind, and at some particular point, are generated recursively from the equation.

Application of (1c) to Several Variables. In conclusion, it should be also noted that (1c) has applications when the $a_m^{(k)}$ are vector, instead of scalar quantities.

One such instance occurs in the conversion of interpolation series to Chebyshev series when we wish to have a general formula for the coefficients in the Chebyshev series in terms of the differences in the interpolation series, instead of just the numerical values of the Chebyshev coefficients corresponding to numerical values of the differences.[9] This is a practical problem when the conversions are for many sets of numerical values of the differences, and a general formula involving no divisions and fewer operations would require less work than the repeated employment of (1c) for each set of numerical differences. To obtain such general formulas, we replace in (1c) the numbers or scalars a_m, $m = 0(1)n$, and $a_m^{(k)}$, $m = 0(1)k$, $k = 0(1)n$, by vectors whose components are numerical coefficients of the various order differences.

Another instance where vectors are employed for a_m and $a_m^{(k)}$ is in the conversion of interpolation series in two variables, say p and q, in terms of differences of mixed order and type involving both variables, expressible as

$$\sum_{m'=0}^{n'} \sum_{m=0}^{n} a_{m,m'} q_m(p) \bar{q}_{m'}(q),$$

into a double Chebyshev series of the form

$$\sum_{m'=0}^{n'} \sum_{m=0}^{n} A_{m,m'} \, T_m^{(a,b)}(p) T_{m'}^{(a',b')}(q).$$

We apply (1c) first in the q-direction, performing $n + 1$ parallel operations, for every vector $a_{m'}^{(k)}$, $m' = 0(1)k$, $k = 0(1)n'$, having $n + 1$ components which are the continually updated coefficients of $q_m(p)$, $m = 0(1)n$, obtaining an expression of the form

$$\sum_{m'=0}^{n'} \left(\sum_{m=0}^{n} b_{m,m'} q_m(p) \right) T_{m'}^{(a',b')}(q).$$

We then apply (1c) in the p-direction, in $n' + 1$ parallel operations to each of $\Sigma_{m=0}^{n} b_{m,m'} q_m(p)$, $m' = 0(1)n'$, obtaining finally $A_{m,m'}$ above.

For more than two variables, the application of (1c) is similar. Thus for three variables, the $a_m^{(k)}$ in (1c) would not be vectors, but two-dimensional matrices. This would occur in the conversion of a difference formula in two variables into a double Chebyshev series whose coefficients are expressed in terms of the differences.

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[9] Cf. [1], which gives the first ten Chebyshev coefficients as functions of the central differences in the Stirling and Bessel formulas through the twelfth degree, for p in $[-1/2, 1/2]$ and $[0, 1]$, respectively.