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Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$. II

By Lowell Schoenfeld

Abstract. In this paper, bounds given in the first part of the paper are strengthened. In addition, it is shown that the interval $(x, x + x/16597)$ contains a prime for all $x \geqslant 2{,}010{,}760$; and explicit bounds for the Chebyshev functions are given under the assumption of the Riemann hypothesis.

We use the references and continue the paragraph numbering of the original paper by Rosser and Schoenfeld [11] and adhere to the same notations except as noted in Section 8 in the case of T_1. New references are given below.

6. Estimates under the Riemann Hypothesis. The result below is of the same strength as that given by von Koch [7] whose estimate used an unspecified constant in place of $1/(8\pi)$.

THEOREM 10. *If the Riemann hypothesis holds, then*

$$|\psi(x) - x|, \quad |\theta(x) - x| < \frac{1}{8\pi}\sqrt{x}\,(\log x - 2)\log x \quad \textit{if} \quad 23 \cdot 10^8 \leqslant x, \tag{6.1}$$

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\,\log^2 x \quad \textit{if} \quad 73.2 \leqslant x, \tag{6.2}$$

$$|\theta(x) - x| < \frac{1}{8\pi}\sqrt{x}\,\log^2 x \quad \textit{if} \quad 599 \leqslant x. \tag{6.3}$$

Also,

$$-\frac{1}{8\pi}\sqrt{x}\,\log^2 x < \psi(x) - x \quad \textit{if} \quad 59 \leqslant x, \tag{6.4}$$

$$\theta(x) - x < \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \textit{if} \quad 0 < x. \tag{6.5}$$

Proof. To handle (6.1), (6.2) and (6.3), we suppose that $x \geqslant \xi \geqslant 82{,}800$. By the Riemann hypothesis and the definitions (3.9) and (3.10), we have $S_3(m, \delta) = 0 = S_4(m, \delta)$. Let

$$\delta = \frac{\log x}{\pi\sqrt{x}} = \frac{\log^2 x}{\pi\sqrt{x}} \cdot \frac{1}{\log x} \leqslant \frac{\alpha_1}{\log x} \leqslant \alpha_2, \tag{6.6}$$

where

$$\alpha_1 = \frac{\log^2 \xi}{\pi\sqrt{\xi}}, \quad \alpha_2 = \frac{\alpha_1}{\log \xi} = \frac{\log \xi}{\pi\sqrt{\xi}} < 0.0126. \tag{6.7}$$

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With T_1 defined by (4.2) and $m = 1$, we have $T_1 > D$; and

$$\log \frac{T_1}{2\pi} + 1 = \log\left\{\frac{\sqrt{x}}{\log x} \cdot \frac{2 + 2\delta + \delta^2}{2 + \delta}\right\} + 1 \leqslant \log\left\{\frac{\sqrt{x}}{\log x}(1 + \delta_1)\right\} + 1$$

$$\leqslant \tfrac{1}{2}\{\log x - 2\log\log x + 2(1 + \delta_1)\} \leqslant \tfrac{1}{2}(\log x - \alpha_3),$$

where

$$(6.8) \qquad \delta_1 = \frac{\alpha_2(1 + \alpha_2)}{2 + \alpha_2} < 0.00634, \qquad \alpha_3 = 2\log\log\xi - 2(1 + \delta_1) > 2.841.$$

We now apply Lemmas 8 and 9 with $m = 1$ to get

$$\frac{1}{x}|\psi(x) - x| < \frac{1}{2\pi\sqrt{x}}\left(1 + \frac{\alpha_1}{2\log x}\right)\left\{\left(\log\frac{T_1}{2\pi} + 1\right)^2 + 1.038207\right\}$$

$$+ \frac{\log x}{2\pi\sqrt{x}} + \frac{1}{x}\left\{\log(2\pi) + \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)\right\}$$

$$< \frac{1}{8\pi\sqrt{x}}\left(1 + \frac{\alpha_1}{2\log x}\right)(\log^2 x - 2\alpha_3 \log x + \alpha_3^2 + 4.152828)$$

$$+ \frac{\log x}{2\pi\sqrt{x}} + \frac{\log(2\pi)}{x}$$

$$(6.9) \qquad \leqslant \frac{1}{8\pi\sqrt{x}}\left(1 + \frac{\alpha_1}{2\log x}\right)(\log^2 x - \alpha_4 \log x) + \frac{\log x}{8\pi\sqrt{x}}\left\{4 + \frac{8\pi\log(2\pi)}{\sqrt{x}\,\log x}\right\},$$

where

$$(6.10) \qquad \alpha_4 = 2\alpha_3 - (\alpha_3^2 + 4.152828)/\log\xi.$$

As $\alpha_3 < 2\log\log\xi < \log\xi$, we see that α_4 increases as α_3 increases; and hence,

$$\alpha_4 > 2(2.841) - (2.841^2 + 4.152828)/\log\xi > 0.$$

Consequently, (6.9) yields

$$(6.11) \qquad |\psi(x) - x|/x < (\log x)(\log x - \alpha_5)/(8\pi\sqrt{x}),$$

where

$$(6.12) \qquad \alpha_5 = \alpha_4 - \alpha_1/2 - 4 - \{8\pi\log(2\pi)\}/(\sqrt{\xi}\log\xi).$$

From [10, (3.39)], we obtain for $x \geqslant \xi \geqslant 82{,}800$

$$(6.13) \qquad \frac{1}{x}|\theta(x) - x| < \frac{1}{x}|\psi(x) - x| + \frac{1}{x}(1.02\sqrt{x} + 3x^{1/3}) < \frac{\log x}{8\pi\sqrt{x}}(\log x - \alpha_6),$$

where

$$(6.14) \qquad \alpha_6 = \alpha_5 - \frac{8.16\pi}{\log\xi} - \frac{24\pi}{\xi^{1/6}\log\xi}.$$

On letting $\xi = 23 \cdot 10^8$, we find that $\alpha_5 > \alpha_6 > 2$ so that (6.1) follows from (6.11) and (6.13).

On letting $\xi = e^{16}$, we find that $\alpha_5 > \alpha_6 > 0$ so that (6.2) and (6.3) hold for all $x \geqslant e^{16}$. If $0 < x < e^{16}$, then [10, Theorem 18], gives $\theta(x) - x < 0$; hence, (6.5) is completely proved. For $1400 \leqslant x < e^{16}$, the same theorem gives

$$\theta(x) - x > -2.05282\sqrt{x} \geqslant -2.05282(\sqrt{x}\log^2 x)/\log^2 1400; \tag{6.15}$$

hence, if $1400 \leqslant x$, then

$$\theta(x) - x > -(\sqrt{x}\log^2 x)/(8\pi). \tag{6.16}$$

For $1200 \leqslant x < 1400$, we use [10, Theorem 19] to replace 2.05282 by 2 in (6.15) and, thereby, derive (6.16) once more. Finally, for $599 \leqslant x < 1200$ we deduce (6.16) from the unpublished Rosser-Walker tables referred to in [10, Section 5]. This completes the proof of (6.3).

As $\psi(x) - x \geqslant \theta(x) - x$, we see that (6.4) holds for $x \geqslant 599$ by (6.3); the proof of (6.4) is completed by using Table VII of Gram [6] for $59 \leqslant x < 599$. Further, [10, Theorems 18 and 24] gives for $0 < x \leqslant 10^8$

$$\psi(x) - x < \psi(x) - \theta(x) < \sqrt{x} + 3x^{1/3} = \sqrt{x}\log^2 x \cdot \left\{ \frac{1}{\log^2 x} + \frac{3}{x^{1/6}\log^2 x} \right\}$$

from which we get for $1{,}075 \leqslant x < e^{16}$

$$\psi(x) - x < (\sqrt{x}\log^2 x)/(8\pi). \tag{6.17}$$

On using Gram's table again, we verify (6.17) for $73.2 \leqslant x < 1{,}075$. This completes the proof of (6.2).

COROLLARY 1. *If the Riemann hypothesis holds, then*

$$|\pi(x) - li(x)| < (\sqrt{x}\log x)/(8\pi) \quad \textit{if } 2{,}657 \leqslant x, \tag{6.18}$$

$$\pi(x) - li(x) < (\sqrt{x}\log x)/(8\pi) \quad \textit{if } 3/2 \leqslant x. \tag{6.19}$$

Proof. Let $x \geqslant 23 \cdot 10^8 \geqslant \xi > 1$. Then [10, (4.17)] yields

$$\pi(x) - \pi(\xi) = \frac{\theta(x)}{\log x} - \frac{\theta(\xi)}{\log \xi} + \int_\xi^x \frac{\theta(y) - y}{y\log^2 y}\,dy + \int_\xi^x \frac{dy}{\log^2 y}. \tag{6.19a}$$

By [10, (7.6)] we get, on putting

$$\xi' = \{li(\xi) - \pi(\xi)\} - \{\xi - \theta(\xi)\}/\log \xi, \tag{6.19b}$$

that for $\xi \geqslant 599$

$$|\pi(x) - li(x)| = \left| \frac{\theta(x) - x}{\log x} + \int_\xi^x \frac{\theta(y) - y}{y\log^2 y}\,dy - \xi' \right|$$

$$\leqslant \frac{1}{8\pi}\sqrt{x}(\log x - 2) + \frac{1}{8\pi}\int_\xi^x \frac{dy}{\sqrt{y}} + |\xi'|$$

$$= \frac{1}{8\pi}\sqrt{x}\log x + |\xi'| - \frac{\sqrt{\xi}}{4\pi}$$

as a result of (6.1) and (6.3). Taking $\xi = 10^8$, we obtain $\xi' \cong 88.26$ so that (6.18) results for $x \geqslant 23 \cdot 10^8$.

It follows from Table 1 of Brent [4] that for all primes $p \leqslant 50 \cdot 10^8$ we have $[li(p) + 1/2] - \pi(p) \leqslant 4612$. If $5 \cdot 10^7 \leqslant x \leqslant 49 \cdot 10^8$ and p is the smallest prime exceeding x, then it is a consequence of Brent [3], [4] that

$$|\pi(x) - li(x)| = \{li(x) - \tfrac{1}{2}\} - \pi(x) + \tfrac{1}{2} < [li(x) + \tfrac{1}{2}] - \pi(x) + \tfrac{1}{2}$$

$$\leqslant [li(p) + \tfrac{1}{2}] - \{\pi(p) - 1\} + \tfrac{1}{2} \leqslant 4613.5 < (\sqrt{x}\log x)/(8\pi).$$

Hence, (6.18) holds for all $x \geqslant 5 \cdot 10^7$. Moreover, the Appel-Rosser table [1961] shows that

$$0 < \{li(x) - \pi(x)\}(\log x)/\sqrt{x} < 2.523 \tag{6.20}$$

if $3{,}169 \leqslant x \leqslant 5 \cdot 10^7$. From this, we get (6.18) for $x \geqslant 3{,}169$. For $2{,}659 \leqslant x < 3{,}169$, the inequality (6.20) holds with 2.523 replaced by 2.444; hence, (6.18) holds for $x \geqslant 2{,}659$. A direct calculation shows that (6.18) holds for $2{,}657 \leqslant x < 2{,}659$.

As $\pi(x) - li(x) < 0$ for $2 \leqslant x \leqslant 2{,}657$ by [10, (4.2)], we easily complete the proof of (6.19).

In the next two corollaries, B, E and C are as defined in [10, Section 2].

COROLLARY 2. *If the Riemann hypothesis holds, then*

$$\left|\sum_{p\leqslant x}\frac{1}{p} - \log\log x - B\right| < \frac{3\log x + 4}{8\pi\sqrt{x}} \qquad \textit{if } 13.5 \leqslant x, \tag{6.21}$$

$$\left|\sum_{p\leqslant x}\frac{\log p}{p} - \log x - E\right| < \frac{3(\log^2 x + 2\log x + 4)}{8\pi\sqrt{x}} \qquad \textit{if } 8.4 \leqslant x. \tag{6.22}$$

Proof. By [10, (2.27)], we obtain for a suitable constant K that

$$\sum_{p\leqslant x}\frac{1}{p} = \int_2^x \frac{dy}{y\log y} + K + \frac{\pi(x) - li(x)}{x} - \int_x^\infty \frac{\pi(y) - li(y)}{y^2}\,dy.$$

By (6.18), we obtain for $x \geqslant 2{,}657$,

$$\left|\sum_{p\leqslant x}\frac{1}{p} - (\log\log x + K - \log\log 2)\right| < \frac{\log x}{8\pi\sqrt{x}} + \frac{1}{8\pi}\int_x^\infty \frac{\log y}{y^{3/2}}\,dy = \frac{3\log x + 4}{8\pi\sqrt{x}}.$$

As the right side tends to 0 as $x \longrightarrow \infty$, $K - \log\log 2$ must be the constant B appearing in [10, Theorem 5]. Now [10, Theorem 20] gives

$$\left|\sum_{p\leqslant x}\frac{1}{p} - (\log\log x + B)\right| < \frac{2}{\sqrt{x}\,\log x} < \frac{3\log x + 4}{8\pi\sqrt{x}}$$

for $32.5 \leqslant x < 2{,}657$. It is then a simple matter to complete the verification of (6.21).

Similarly, [10, (2.27)] gives

$$\sum_{p\leqslant x}\frac{\log p}{p} = \int_2^x \frac{dy}{y} + K^* + \frac{\pi(x) - li(x)}{x}\log x + \int_x^\infty \frac{1 - \log y}{y^2}\{\pi(y) - li(y)\}\,dy.$$

By (6.18), we get for $x \geqslant 2{,}657$

$$\left|\sum_{p\leqslant x}\frac{\log p}{p}-(\log x+E)\right|<\frac{\log^2 x}{8\pi\sqrt{x}}+\frac{1}{8\pi}\int_x^\infty\frac{(\log y-1)\log y}{y^{3/2}}\,dy.$$

This yields (6.22) for $x \geqslant 2{,}657$. For $16.1 \leqslant x < 2{,}657$ we obtain the result by using [10, Theorem 21]. Direct verification completes the proof of (6.22).

COROLLARY 3. *If the Riemann hypothesis holds, then*

$$\left|e^C(\log x)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)-1\right|<\frac{3\log x+5}{8\pi\sqrt{x}}\quad \textit{if } 8.0\leqslant x, \tag{6.23}$$

$$\left|\frac{e^{-C}}{\log x}\prod_{p\leqslant x}\frac{p}{p-1}-1\right|<\frac{3\log x+5}{8\pi\sqrt{x}}\quad \textit{if } 13.1\leqslant x. \tag{6.24}$$

Proof. Let

$$y=\frac{3\log x+4}{8\pi\sqrt{x}},\qquad z=y+\frac{1}{8\pi\sqrt{x}},\qquad S=\sum_{p>x}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\} \tag{6.25}$$

so that this definition of S agrees with that below [10, (8.10)] where it is proved that

$$0>S>-\frac{1.02}{(x-1)\log x}\equiv -S_0\quad \text{if } 1<x. \tag{6.26}$$

By (6.21), if $x \geqslant 13.5$ there is a $\vartheta=\vartheta_x\in(-1,1)$ such that

$$\log\log x+\vartheta y=\sum_{p\leqslant x}\frac{1}{p}-B=-\sum_{p\leqslant x}\log\left(1-\frac{1}{p}\right)-S-C$$

as a result of [10, (2.7)]. Hence,

$$e^C(\log x)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)=e^{-\vartheta y-S}. \tag{6.27}$$

We easily verify that $y+S_0<2.4\cdot 10^{-4}$ if $x\geqslant 10^8$ so that

$$\exp(-\vartheta y-S)\leqslant\exp(y+S_0)\leqslant 1+(y+S_0)+0.501(y+S_0)^2<1+z. \tag{6.28}$$

Hence, (6.27) gives for $x\geqslant 10^8$

$$e^C(\log x)\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)<1+z. \tag{6.29}$$

By [10, Theorem 23], we see that this holds for all $x>1$. As a result, for $x>1$,

$$\frac{e^{-C}}{\log x}\prod_{p\leqslant x}\frac{p}{p-1}>\frac{1}{1+z}>1-z. \tag{6.30}$$

Similarly, if $x\geqslant 10^8$ then (6.27) and (6.28) give

$$\frac{e^{-C}}{\log x}\prod_{p\leqslant x}\frac{p}{p-1}=e^{\vartheta y+S}<e^y<e^{y+S_0}<1+z. \tag{6.31}$$

Also, [10, Theorem 23] gives

$$\frac{e^{-C}}{\log x}\prod_{p\leqslant x}\frac{p}{p-1}<1+\frac{2}{\sqrt{x}\log x}<1+z,$$

provided $28.4\leqslant x<10^8$. Moreover, the extreme inequality in (6.31) is easily seen to hold for $13.1\leqslant x<28.4$ as well; and this verifies (6.24) when use is made of (6.30). As (6.31) holds for all $x\geqslant 13.1$, we have

$$e^C(\log x) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) > \frac{1}{1+z} > 1 - z$$

for $x \geqslant 13.1$. Again, the extreme inequality holds for $8.0 \leqslant x < 13.1$ so that (6.23) follows from (6.29).

7. **Bounds for Large** x. The following result improves both Theorems 2 and 3 Moreover, even better results are given by the Corollary to Theorem 11.

THEOREM 11. *Let* $X = \sqrt{(\log x)/R}$ *where* $R = 9.6459\ 08801$, *and let*

$$\epsilon_0(x) = \sqrt{8/17\pi}\ X^{1/2} e^{-X}. \tag{7.1}$$

Then,

$$|\psi(x) - x| < x\,\epsilon_0(x) \quad \text{if} \quad 17 \leqslant x, \tag{7.2}$$

$$|\theta(x) - x| < x\,\epsilon_0(x) \quad \text{if} \quad 101 \leqslant x, \tag{7.3}$$

$$\theta(x) - x \leqslant \psi(x) - x\,\epsilon_0(x) \quad \text{if} \quad 1 \leqslant x. \tag{7.4}$$

Proof. The main part of the proof is concerned with large x in which case the proof is similar to that given for Theorem 3, but we ultimately take $m = 2$ rather than $m = 1$. In place of (3.36), we let

$$T_2 = 17e^{vX}, \tag{7.5}$$

where v will be specified later. We assume that v, m, X are such that

$$A \leqslant T_2, \qquad 1/\sqrt{m+1} \leqslant v \leqslant 1, \tag{7.6}$$

from which we deduce $X \geqslant \log(A/17) > 11.62$ and $W_m \leqslant T_2 \leqslant W_0$ by (3.24).

In place of (3.37), we get

$$S_3(m, \delta) \leqslant \frac{2 + m\delta}{2} \left(\left\{ \frac{1}{2\pi} - q(T_2) \right\} \int_A^{T_2} \phi_0(y) \log \frac{y}{2\pi}\, dy + E_1 \right), \tag{7.7}$$

where (3.2) gives

$$E_1 = \{N(T_2) - F(T_2) + R(T_2)\}\phi_0(T_2) - \{N(A) - F(A) + R(A)\}\phi_0(A)$$

$$< 2R(T_2)\,\phi_0(T_2), \tag{7.8}$$

and $R(T) = 0.137 \log T + 0.443 \log\log T + 1.588$ as in Rosser [1941]. Putting $V'' \doteq X^2/\log(T_2/17)$, we have

$$V'' = X/v = X\{2 - v + (1 - v)^2/v\} = Y + 2X - vX, \tag{7.9}$$

where

$$Y = X(1 - v)^2/v. \tag{7.10}$$

Proceeding as in (3.38) and (3.41) and using (7.8), we find

$$S_3(m, \delta) < \frac{2 + m\delta}{2} \cdot \frac{1}{2\pi}\, e^{-V''}\{X^4(V'')^{-3} + X^2 d(V'')^{-2}\} + \frac{2 + m\delta}{2}\, E_1$$

$$< \frac{2 + m\delta}{68\pi}\, G_0 e^{-Y} X e^{-2X} T_2 + (2 + m\delta) R(T_2)\phi_0(T_2), \tag{7.11}$$

where $d = \log(17/2\pi) = 0.99533\ldots$ and

$$G_0 = v^2(v + d/X). \tag{7.12}$$

As $R(y)/\log y$ decreases for $y > e^e$, we have

$$R(T_2)\phi_0(T_2) = \frac{R(T_2)}{\log T_2}\,\phi_0(T_2)\log T_2 \leqslant \frac{R(A)}{\log A}\cdot\frac{\log T_2}{T_2}\,e^{-V''}$$

$$= \frac{R(A)}{17\log A}\,e^{-Y}e^{-2X}\log T_2$$

by (7.9). Then (7.5) and (7.6) yield

$$\log T_2 = vX + \log 17 \leqslant X + \log 17 < 1.244X;$$

hence,

$$R(T_2)\phi_0(T_2) < 0.0241e^{-Y}Xe^{-2X}. \tag{7.13}$$

In place of (3.42), we have

$$S_4(m, \delta) \leqslant R_m(\delta)\delta^{-m}\left(\left\{\frac{1}{2\pi} + q(T_2)\right\}\int_{T_2}^{\infty}\phi_m(y)\log\frac{y}{2\pi}\,dy + E_0\right), \tag{7.14}$$

where, by (3.2),

$$E_0 = \{R(T_2) + F(T_2) - N(T_2)\}\,\phi_m(T_2)$$

$$< 2R(T_2)\phi_m(T_2) = 2R(T_2)\phi_0(T_2)T_2^{-m}. \tag{7.15}$$

By (3.16),

$$\int_{T_2}^{\infty}\phi_m(y)\log\frac{y}{2\pi}\,dy = \frac{z^2}{2m^2 17^m}\left\{K_2(z, U') + \frac{2dm}{z}\,K_1(z, U')\right\}, \tag{7.16}$$

where we put $z = 2X\sqrt{m}$ and

$$U' = (2m/z)\log(T_2/17) = v\sqrt{m}.$$

We strengthen part of (7.6) by assuming

$$v > 1/\sqrt{m} \tag{7.17}$$

so that $U' > 1$; also $m \geqslant 2$ since $1 \geqslant v$ by (7.6).

By Lemma 4 and the Corollary of Lemma 5,

$$K_2(z, U') + \frac{2dm}{z}\,K_1(z, U') < \left(U' + \frac{2}{z} + \frac{2dm}{z}\right)Q_1(z, U')$$

$$\leqslant \sqrt{m}\left(v + \frac{1 + dm}{mX}\right)\frac{U'^2}{z(U'^2 - 1)}\,\exp\left\{-\frac{z}{2}\left(U' + \frac{1}{U'}\right)\right\}.$$

Now

$$\frac{z}{2}\left(U' + \frac{1}{U'}\right) = X\sqrt{m}\left\{v\sqrt{m} + \frac{1}{v\sqrt{m}}\right\} = mvX + (Y + 2X - vX)$$

by (7.9). Hence,

$$K_2(z, U') + \frac{2dm}{z}\,K_1(z, U') < G_1 e^{-Y}\,\frac{m}{2(m-1)}\,X^{-1}e^{-2X}\left(\frac{T_2}{17}\right)^{-(m-1)}, \tag{7.18}$$

where

$$G_1 = \frac{m-1}{m} \cdot \frac{U'^2}{U'^2-1}\left(v + \frac{1+dm}{mX}\right) = \frac{(m-1)v^2}{mv^2-1}\left(v + \frac{1+dm}{mX}\right). \tag{7.19}$$

Then (7.16) and (7.18) give

$$\int_{T_2}^{\infty} \phi_m(y)\log\frac{y}{2\pi}\,dy < \frac{G_1 e^{-Y}}{17(m-1)}\; Xe^{-2X}T_2^{-(m-1)}.$$

We define

$$G_2 = \frac{R_m(\delta)}{2^m}\{1 + 2\pi q(T_2)\} = \{1 + 2\pi q(T_2)\}\left\{\frac{(1+\delta)^{m+1}+1}{2}\right\}^m. \tag{7.20}$$

Then (7.14) yields

$$S_4(m,\delta) < \frac{G_2 G_1 e^{-Y}}{34\pi(m-1)}\left(\frac{2}{\delta}\right)^m Xe^{-2X}T_2^{-(m-1)} + G_2\left(\frac{2}{\delta}\right)^m E_0.$$

Now $1 + m\delta/2 < R_m(\delta)/2^m < G_2$. Using (7.11) and (7.15), we obtain

$$S_3(m,\delta) + S_4(m,\delta) < \frac{G_2 e^{-Y}}{34\pi}\cdot Xe^{-2X}\left\{G_0 T_2 + \frac{G_1}{m-1}\left(\frac{2}{\delta}\right)^m T_2^{-(m-1)}\right\}$$

$$+ 2G_2 R(T_2)\phi_0(T_2)\left\{1 + \left(\frac{2}{\delta T_2}\right)^m\right\}.$$

If G_0 and G_1 were independent of v, and hence of T_2, then the expression inside the first braces would be minimized by choosing

$$T_2 = (G_1/G_0)^{1/m}\cdot 2/\delta. \tag{7.21}$$

Postponing the reconciliation of this with (7.5), we obtain

$$S_3(m,\delta) + S_4(m,\delta) + \frac{1}{2}m\delta < \frac{1}{2}mG_2\left\{G_0^{1-1/m}G_1^{1/m}\frac{2e^{-Y}}{17\pi(m-1)}\; Xe^{-2X}\delta^{-1} + \delta\right\}$$

$$+ 2G_2(1 + G_0/G_1)R(T_2)\phi_0(T_2).$$

The expression inside the last braces is minimized by choosing

$$\delta = \left\{G_0^{1-1/m}G_1^{1/m}\frac{2e^{-Y}}{17\pi(m-1)}\right\}^{1/2} X^{1/2}e^{-X} \tag{7.22}$$

so that (7.21) becomes

$$T_2 = (G_1/G_0)^{1/2m}\{34\pi(m-1)e^{Y}/G_0\}^{1/2}X^{-1/2}e^{X}. \tag{7.23}$$

Moreover, (7.13) gives

$$S_3(m,\delta) + S_4(m,\delta) + \frac{1}{2}m\delta < G_2\left\{G_0^{1-1/m}G_1^{1/m}\frac{2e^{-Y}}{17\pi}\right\}^{1/2}\frac{m}{\sqrt{m-1}}\; X^{1/2}e^{-X}$$

$$+ 0.0482\,G_2(1 + G_0/G_1)e^{-Y}Xe^{-2X}.$$

The coefficient $m/\sqrt{m-1}$ in the next to the last term is minimized by choosing $m = 2$. For this value, we obtain from (7.22), (7.23) and (7.19), (7.24a) $$\delta = (G_0G_1)^{1/4}e^{-Y/2}\sqrt{2/17\pi}\,X^{1/2}e^{-X},$$

(7.24b) $$T_2 = (G_1/G_0^3)^{1/4}e^{Y/2}\sqrt{34\pi}\,X^{-1/2}e^{X},$$

(7.25) $$G_1 = v^2\{v + (1 + 2d)/(2X)\}/(2v^2 - 1).$$

Also,

(7.26) $$\begin{aligned} S_3(2,\delta) + S_4(2,\delta) + \delta &< G_2(G_0G_1)^{1/4}e^{-Y/2}\sqrt{8/17\pi}\,X^{1/2}e^{-X} \\ &\quad + 0.0482G_2(1 + G_0/G_1)e^{-Y}Xe^{-2X} \end{aligned}$$

provided the choice of T_2 in (7.24b) is consistent with (7.5) and provided both (7.6) and (7.17) hold when $m = 2$.

We readily see that the T_2 of (7.24b) satisfies (7.5) if and only if v is such that $k(v) = 1$ where

(7.27) $$k(v) = \frac{17X}{2\pi}\left(\frac{G_0^3}{G_1}\right)^{1/2} e^{-2X(1-v)}e^{-X(1-v)^2/v}$$

and (7.10) has been used. If $1/\sqrt{2} < v \leqslant \sqrt{3/2}$, it is not hard to see that G_1 decreases as v increases. By (7.12), it then follows that $k(v)$ is strictly increasing for increasing $v \in (1/\sqrt{2}, 1]$. Now $k(v) \longrightarrow 0$ as $v \longrightarrow 1/\sqrt{2}$ from the right; and we easily see that $k(1) > 1$ (for all $X \geqslant 1$). As a result, there is a unique $v \in (1/\sqrt{2}, 1)$ such that $k(v) = 1$. Henceforth, let v be this number so that v depends on X; then G_0, G_1, Y and T_2 are defined in terms of v by (7.12), (7.25), (7.10), and (7.5), (7.24b). Of course, (7.17) holds since $m = 2$. Hence, (7.26) will be fully established once it is shown that $T_2 \geqslant A$.

We have, for $1/\sqrt{2} < v \leqslant 1$,

(7.28) $$H(v) \equiv \frac{G_0^3}{G_1} = v^4(2v^2 - 1)\frac{(v + d/X)^3}{v + (1 + 2d)/(2X)}\begin{cases} < (v + d/X)^2 \\ > v^6(2v^2 - 1). \end{cases}$$

If we define, for $j = 0$ and 1,

(7.29) $$v_j = 1 - \frac{1}{2X}\log\frac{17X}{(2 + 3j)\pi},$$

then $H(v_0) < 1$ if $X \geqslant 17/(2\pi)$; also, $H(v_1) > 0.22318$ if $X \geqslant 8.579$. Inasmuch as

$$k(v_j) = \frac{2 + 3j}{2}\sqrt{H(v_j)}\exp\left\{-\frac{1}{4v_jX}\log^2\frac{17X}{(2 + 3j)\pi}\right\},$$

we see that $k(v_0) < 1 = k(v)$ if $X \geqslant 17/(2\pi)$; also $k(v_1) > 1 = k(v)$ if $X \geqslant 8.579$. So

(7.30) $$v_0 < v \ \text{ if } \log x \geqslant 71; \qquad v < v_1 \ \text{ if } \log x \geqslant 710.$$

Of course, $v_0 < v_1$ in all cases. For $\log x \geqslant 1737$, we now get from (7.5) and (7.29) that $T_2 > 17e^{v_0X} > A$.

Hence, (7.26) is completely established when $\log x \geqslant 1737$; for these x, we have $v_0 > 0.8661 > \sqrt{3/4}$. It is a simple matter to use (7.12), (7.25), (7.10), (7.30) and (7.29) to verify that

(7.31) $$G_0/G_1 < 2v^2 - 1 < 1, \quad Y < X(1 - v_0)^2/v_0 < 0.278,$$

$$(7.32)\quad G_0G_1 = \frac{v^6}{2v^2-1}\left(1+\frac{d}{vX}\right)\left(1+\frac{1+2d}{2vX}\right)\begin{cases} < (1+1/vX)(1+3/2vX) < 1.227 \\ > v^6/(2v^2-1) > 0.843 \end{cases}$$

for $\log x \geqslant 1737$. Then (7.26) yields

$$(7.33)\quad S_3(2,\delta)+S_4(2,\delta)+\delta < G_2(G_0G_1)^{1/4}e^{-Y/2}\left\{\sqrt{\frac{8}{17\pi}}\,X^{1/2}e^{-X} + 0.11Xe^{-2X}\right\}.$$

Taking $T_1 = 0$ in (3.7) and (3.8) and using Lemma 17 of Rosser [1941], we obtain

$$\frac{1}{\sqrt{x}}\{S_1(2,\delta)+S_2(2,\delta)\} \leqslant \frac{1}{\sqrt{x}}\cdot\frac{R_2(\delta)}{\delta^2}\sum_\gamma \frac{1}{|\gamma|^3} < \frac{1}{\sqrt{x}}\,G_2\left(\frac{2}{\delta}\right)^2\frac{1}{14.13}\sum_\gamma \frac{1}{\gamma^2}$$
$$< \frac{4}{\delta^2\sqrt{x}}\,G_2\,\frac{0.0463}{14.13} < 0.61\,G_2(G_0G_1)^{1/4}e^{-Y/2}X^{-1}e^{2X}x^{-1/2},$$

by (7.24a), (7.31) and (7.32). Putting

$$\Omega = \{S_1(2,\delta)+S_2(2,\delta)\}/\sqrt{x} + S_3(2,\delta)+S_4(2,\delta)+\delta,$$

we obtain from (7.33) that for $\log x \geqslant 1737$

$$(7.34)\quad \Omega < G_2(G_0G_1)^{1/4}e^{-Y/2}\left\{\sqrt{\frac{8}{17\pi}}\,X^{1/2}e^{-X} + 0.11\,Xe^{-2X} + 0.61\,X^{-1}e^{2X}x^{-1/2}\right\}.$$

By Lemma 8

$$\frac{1}{x}|\psi(x)-x| < \frac{1}{x}\left\{\log(2\pi)+\frac{1}{2}\log(1-x^{-2})\right\} + \Omega < \Omega + \frac{\log(2\pi)}{x}.$$

Now [10, Theorem 13] gives

$$\frac{1}{x}|\theta(x)-x| < \Omega + \frac{\log(2\pi)}{x} + \frac{1.43}{\sqrt{x}}$$
$$< \Omega + 0.01\,G_2(G_0G_1)^{1/4}e^{-Y/2}\cdot X^{-1}e^{2X}x^{-1/2}.$$

Hence, (7.34) and (7.1) give

$$(7.35)\quad \frac{1}{x}|\psi(x)-x|,\quad \frac{1}{x}|\theta(x)-x| < G_3\sqrt{\frac{8}{17\pi}}\,X^{1/2}e^{-X} = G_3\epsilon_0(x)$$

for $\log x \geqslant 1737$, where

$$G_3 = G_2(G_0G_1)^{1/4}e^{-Y/2}\left\{1+\sqrt{\frac{17\pi}{8}}\,(0.11X^{1/2}e^{-X} + 0.62\,X^{-3/2}e^{3X}x^{-1/2})\right\}$$
$$< G_2(G_0G_1)^{1/4}e^{-Y/2}\{1+0.29\,X^{1/2}e^{-X}\}$$
$$(7.36)\quad < G_2(G_0G_1)^{1/4}e^{-Y/2}\exp(0.29X^{1/2}e^{-X}).$$

Also, by the definition of $q(y)$, (7.24b) and (7.28)

$$1 + 2\pi q(T_2) = 1 + \frac{2\pi}{T_2} \cdot \frac{0.137 + 0.443/\log T_2}{\log(T_2/2\pi)}$$

$$< 1 + \sqrt{\frac{2\pi}{17}} \left(\frac{G_0^3}{G_1}\right)^{1/4} e^{-Y/2} \, \frac{0.137 + 0.443/\log A}{\log(A/2\pi)} \, X^{1/2} e^{-X}$$

$$< 1 + 0.01\, X^{1/2} e^{-X} < \exp(0.01\, X^{1/2} e^{-X}).$$

Further,

$$\frac{R_2(\delta)}{2^2} = \left\{\frac{(1+\delta)^3 + 1}{2}\right\}^2 = \left\{1 + \frac{1}{2}\delta(3 + 3\delta + \delta^2)\right\}^2 < \left(1 + \frac{3.01}{2}\,\delta\right)^2$$

$$< \left\{\exp\left(\frac{3.01}{2}\,\delta\right)\right\}^2 = \exp(3.01\,\delta) < \exp(0.62 X^{1/2} e^{-X}).$$

Then (7.36) and (7.20) give

$$G_3 < (G_0 G_1)^{1/4} e^{-Y/2} \exp(0.92\, X^{1/2} e^{-X}) = \{G_0 G_1 e^{-2Y} \exp(3.68\, X^{1/2} e^{-X})\}^{1/4}.$$

By (7.12), we obtain for $\log x \geqslant 1737$,

$$\frac{X}{v^2} G_0 \exp(3.68 X^{1/2} e^{-X}) < X(v + d/X)(1 + 3.69 X^{1/2} e^{-X})$$

$$= Xv + d + (Xv + d) 3.69 X^{1/2} e^{-X}$$

$$< Xv + d + 0.0003 < X(v + 1/X).$$

Hence, (7.25) yields

$$G_3 < \left\{v^2 \left(v + \frac{1}{X}\right) G_1 e^{-2Y}\right\}^{1/4} < \left\{\frac{v^4}{2v^2 - 1}\left(v + \frac{1}{X}\right)\left(v + \frac{3}{2X}\right) e^{-2Y}\right\}^{1/4}.$$

As a result of (7.35), we deduce for $\log x \geqslant 1737$,

(7.37) $$|\psi(x) - x|, \quad |\theta(x) - x| < x\epsilon_0(x) M(v) L(v),$$

where

(7.38) $$L(v) = \{v^6/(2v^2 - 1)\}^{1/4},$$

(7.39) $$M(v) = \{(1 + 1/vX)(1 + 3/2vX) e^{-2X(1-v)^2/v}\}^{1/4}.$$

The function $L(v)$ is real-valued for $v > 1/\sqrt{2}$ and, as is easily seen, has a minimum value at $v = \sqrt{3/4}$. If $\log x \geqslant 164$, then $v > v_0 > 1/\sqrt{2}$ by (7.29). Also, if $\log x \geqslant 448$, then $v > v_0 > 0.78617$ by (7.29). Hence,

(7.40) $$L(v) > (27/32)^{1/4} \quad \text{if } x \geqslant e^{164}; \qquad L(v) < 1 \quad \text{if } x \geqslant e^{448}.$$

In addition, (7.30) and (7.29) yield for $\log x \geqslant 710$

$$M(v) < \exp \frac{1}{4}\left\{\frac{1}{vX} + \frac{3}{2vX} - \frac{2X}{v}(1 - v_1)^2\right\}$$

(7.41) $$= \exp\left\{\frac{-1}{8vX}\left(\log^2 \frac{17X}{5\pi} - 5\right)\right\} < E(x),$$

where

$$E(x) = \exp\left\{\frac{-1}{8v_1X}\left(\log^2\frac{17X}{5\pi} - 5\right)\right\} = \exp\frac{1}{4v_1}\left\{\frac{5}{2X} - 2X(1-v_1)^2\right\}. \tag{7.42}$$

It is clear from the first part of (7.42) that $E(x) < 1$ if $\log x \geqslant 721$. By (7.37), (7.40) and (7.41), it follows that (7.2) and (7.3) hold for all $x \geqslant e^{1737}$.

Next, we prove the following strengthened form of (7.2) and (7.3) for $10^8 \leqslant x < e^{1737}$:

$$|\psi(x) - x|, \quad |\theta(x) - x| < 0.802\, x\epsilon_0(x). \tag{7.43}$$

By [10, (3.36)], we have

$$0 \leqslant \psi(x) - \theta(x) < 1.427\sqrt{x} \leqslant 0.00015x \quad \text{if} \quad 10^8 \leqslant x.$$

The table in Section 5 then shows that for $10^8 \leqslant x$,

$$|\theta(x) - x| \leqslant |\psi(x) - x| + \psi(x) - \theta(x) < 0.00121x + 0.00015x = 0.00136x.$$

As a result, if $10^8 \leqslant x \leqslant e^{350}$ then

$$|\theta(x) - x|, \quad |\psi(x) - x| < 0.00136x\,\frac{\epsilon_0(x)}{\epsilon_0(e^{350})} < \frac{0.00136}{0.00229}\, x\epsilon_0(x) < 0.594x\epsilon_0(x).$$

Similarly, if $x > e^{350}$ then $\psi(x) - \theta(x) < 10^{-75}x$ so that for $e^{350} < x \leqslant e^{1200}$ the table yields

$$|\theta(x) - x|, \quad |\psi(x) - x| < \frac{1.42 \cdot 10^{-5}}{1.85 \cdot 10^{-5}}\, x\epsilon_0(x) < 0.768x\epsilon_0(x).$$

We continue in this way using the table in Section 5 for $b = 1200, 1400, 1500, 1600$ and the table below for $b = 1650$ and $b = 1700$; we thereby prove (7.43), and, hence, (7.2) and (7.3) for $10^8 \leqslant x < e^{1737}$.

For smaller x, we proceed as in the proof of Theorem 9.* Inasmuch as $\epsilon_0(x)$ increases for $0 < X < 1/2$ and decreases for $X > 1/2$, we have that

$$\epsilon_0(x) \geqslant \min\{\epsilon_0(2), \epsilon_0(10^8)\} > 0.11 \quad \text{if} \quad 2 \leqslant x \leqslant 10^8. \tag{7.44}$$

Now [10, Theorem 10] gives

$$\theta(x) > 0.89x > x - x\epsilon_0(x) \quad \text{if} \quad 227 \leqslant x < 10^8.$$

If $149 \leqslant x < 227$ then $\epsilon_0(x) > 0.15$, and if $101 \leqslant x < 139$ then $\epsilon_0(x) > 0.16$; applying [10, Theorem 10], we obtain

$$\theta(x) - x > -x\epsilon_0(x) \quad \text{if} \quad x \geqslant 101 \tag{7.45}$$

except for $139 \leqslant x < 149$. For these x, we have $\epsilon_0(x) > 0.159$ and $\theta(x) > 126 > 0.845x > x - x\epsilon_0(x)$ so that (7.45) is completely proved. As a consequence of this and an easy verification for $17 \leqslant x < 101$, we get

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*Note that in (5.10), the correct range for x is given by $1 \leqslant x$, but in (5.11) the correct range is $41 \leqslant x$.

$$\psi(x) > x - x\epsilon_0(x) \quad \text{for} \quad x \geqslant 17. \tag{7.46}$$

Moreover, [10, (3.35)] and (7.44) give

$$\theta(x) \leqslant \psi(x) < 1.04x < x + x\epsilon_0(x) \quad \text{if} \quad 2 \leqslant x \leqslant 10^8.$$

For $1 \leqslant x < 2$, we have $\psi(x) = 0 < x + x\epsilon_0(x)$. On using (7.46) and (7.45), we obtain the complete proof of (7.4), (7.2) and (7.3).

COROLLARY. *If* v_1 *is defined by* (7.29) *and* $v \in (1/\sqrt{2}, 1)$ *is the unique solution of* $k(v) = 1$, *then*

$$|\psi(x) - x|, \quad |\theta(x) - x| < x\epsilon_0(x)E(x)L(v_1) \quad \textit{if} \quad e^{710} \leqslant x, \tag{7.47}$$

$$|\psi(x) - x|, \quad |\theta(x) - x| < x\epsilon_0(x)M(v)L(v) \quad \textit{if} \quad e^{687} \leqslant x. \tag{7.48}$$

Proof. If $\log x \geqslant 687$, then (7.39), (7.30) and (7.29) give

$$M(v) > \exp\left\{\frac{X(1 - v_0)^2}{2v_0}\right\} = \exp\left\{\frac{-1}{8v_0X}\log^2\frac{17X}{2\pi}\right\} > 0.837.$$

For arbitrary $v' > 1/\sqrt{2}$ and $687 \leqslant \log x < 1737$, we obtain from (7.43) and (7.40)

$$\begin{aligned} |\psi(x) - x|, \quad |\theta(x) - x| &< 0.802\,x\epsilon_0(x)\{M(v)/0.837\}\,\{L(v')(27/32)^{-1/4}\} \\ &< x\epsilon_0(x)M(v)L(v'). \end{aligned} \tag{7.49}$$

We use (7.41) and $v_1 > 1/\sqrt{2}$ to get (7.47) for $710 \leqslant \log x < 1737$; the proof is completed by using (7.37), (7.41) and (7.30) which imply $\sqrt{3/4} < v < v_1$. As $v > v_0 > 1/\sqrt{2}$ for $\log x \geqslant 1$ (7.49) (7.30), we deduce (7.48) from (7.49) and (7.37).

Thus, apart from the most easily computed bound $x\epsilon_0(x)$ given by (7.2) and (7.3), we have the more precise bounds of (7.47) and (7.48). Of the latter two, (7.48) provides a tighter bound, but it is more difficult to compute because of the effort required to solve $k(v) = 1$ for v. In the next section, we make further remarks on these bounds.

We also note that the range for x can be extended in (7.47) and (7.48), but there is no point in doing so because (7.43) is better as is the table, Also, by a more careful treatment of the estimates for $\Gamma(-2, V'')$, $\Gamma(-1, V'')$, $K_2(z, U')$, $K_1(z, U')$ in the work leading to (7.11) and (7.18), we could derive a version of (7.37) with a slightly smaller $M(v)$.

This Theorem 11 provides better results than both Theorems 2 and 3. For, $\epsilon_0(x) < \epsilon(x)$ if $1 < x$ as a consequence of the fact that

$$\epsilon(x)/\epsilon_0(x) = 0.257634(8/17\pi)^{-1/2}(X^{1/4} + 0.96642X^{-3/4}) > 1,$$

as we see by evaluating the expression in the middle for $X = 3(0.96642)$ where it assumes its minimum value. It follows from a remark just below (3.35) that $\epsilon_0(x) < \epsilon(x) < \epsilon^*(x)$ for $0 < X \leqslant 59$; and if $X > 59$, then

$$\epsilon_0(x) = \sqrt{8/17\pi}\;X^{-1/4} \cdot X^{3/4}e^{-X} < 0.14X^{3/4}e^{-X} < \epsilon^*(x).$$

Thus, $\epsilon_0(x) < \epsilon^*(x)$ for all $x > 1$.

It is clear from the proof of Theorem 11 that, with the present methods of estimating the various sums and integrals, the choice $m = 2$ is optimal although the main portion of the paper contains statements that might appear to suggest that $m = 1$ is the best choice.

It is interesting to note that for large x each of the terms $S_3(2, \delta)$ and $S_4(2, \delta)$ contributes about $\frac{1}{2}\delta \sim \frac{1}{4}\epsilon_0(x)$; the remaining contribution comes from $\frac{1}{2}m\delta = \delta \sim \frac{1}{2}\epsilon_0(x)$ so that the three terms indicated contribute a total of about $\epsilon_0(x)$. We also note that $\delta \longrightarrow 0$ as $x \longrightarrow \infty$ and that $\delta T_2 \longrightarrow 2$.

We remark that incomplete Bessel functions have been studied in the book of Agrest and Maksimov [1]. Our function $K_\nu(z, x)$ of (2.1) does not appear in this work which assigns a different meaning to this symbol on page 26 of the English translation. See Binet [2], where $K_{1/2}(z, x)$ is expressed as the sum of two terms involving the complementary error function; cf. the work above beginning at (2.20). Also, Faxén [5] gives series expansions in ascending powers of z (which are not useful for our purposes).

8. **Numerical Bounds for Moderate Values of** x. In this section, we show how the results of Section 4 can be improved. One source of improvement is through the replacement of the D of (4.3) by a larger value. Another results from using closer approximations to $\Gamma(\nu, x)$ in Theorem 5 than those given in (4.12) and (4.13). A third source stems from the selection of $m = 2$, rather than $m = 1$, for large b, coupled with the proper choice of T_2 rather than that given by (3.36).

To facilitate the discussion of the latter point, let us define

$$T_0 = \frac{1}{\delta}\left(\frac{2R_m(\delta)}{2 + m\delta}\right)^{1/m}; \tag{8.1}$$

earlier, in (4.2), the quantity on the right was called T_1, but we now leave T_1 unspecified for the moment. It is clear from (7.7), (7.14), (7.8), (7.15) and (7.5) that

$$S_3(m, \delta) + S_4(m, \delta) < h_3(T_2)/(2\pi) + e_3(T_2), \tag{8.2}$$

where

$$h_3(T) = \frac{2 + m\delta}{2}\int_A^T \phi_0(y)\log\frac{y}{2\pi}\,dy + R_m(\delta)\delta^{-m}\int_T^\infty \phi_m(y)\log\frac{y}{2\pi}\,dy,$$

$$e_3(T) = q(T)\left\{-\frac{2 + m\delta}{2}\int_A^T \phi_0(y)\log\frac{y}{2\pi}\,dy + R_m(\delta)\delta^{-m}\int_T^\infty \phi_m(y)\log\frac{y}{2\pi}\,dy\right\}$$
$$+ R(T)\phi_0(T)\{2 + m\delta + 2R_m(\delta)(\delta T)^{-m}\}.$$

As $e_3(T_2)$ corresponds to the last term in (7.26), it is easy to see that it is small compared to $h_3(T_2)$ so that we can approximately minimize the right side of (8.2) by minimizing $h_3(T_2)$. (If, in place of (7.6), we only assumed that $A \leqslant T_2 \leqslant W_0$ as in Theorem 5, then the form of $e_3(T)$ changes, but it is still small compared with $h_3(T)$.) We have

$$h_3'(T) = \frac{2+m\delta}{2}\,\phi_0(T)\log\frac{T}{2\pi} - R_m(\delta)\delta^{-m}\phi_m(T)\log\frac{T}{2\pi}$$

$$(8.3)\qquad = \phi_0(T)\log\frac{T}{2\pi}\cdot\left\{\frac{2+m\delta}{2} - R_m(\delta)(\delta T)^{-m}\right\},$$

and this is clearly negative, zero or positive for $T > 2\pi$ according as $T < T_0$, $T = T_0$ or $T > T_0$. Consequently, $h_3(T)$ is minimal for $T = T_0$. Hence, in Theorem 5, we should choose $T_2 = T_0$ provided $T_0 \geqslant A$. If the last condition is not satisfied, then Theorem 4, corresponding to $T_0 = 0$ (or, equivalently, $T_0 = A$), should be used.

We note from (7.21) that in Theorem 11 we had defined T_2 and δ in such a way that $T_2 = (G_1/G_0)^{1/m}\cdot 2/\delta$ which does not exactly coincide with the optimal choice T_0 of (8.1). The reason for this is that T_2 essentially minimizes an upper bound for $h_3(T)$ whereas T_0 minimizes $h_3(T)$ itself. Nevertheless, (8.1) and (3.6) show that $\delta T_0 \longrightarrow 2$ as $\delta \longrightarrow 0$ so that $T_0 \sim 2/\delta \sim T_2$ as $\delta \longrightarrow 0$; this confirms that the choice of T_2 in Theorem 11 is asymptotically best. However, for the δ and T_2 of (3.35) and (3.36), we have $T_2 \sim c_0/\delta$ where c_0 is just about $(17/\sqrt{\pi})^{1/2} \cong 3.097$; hence, the T_2 of (3.36) is about 50% too large and should not have been used in calculating the last three entries in the table on page 267 of the main part of this paper.

The situation for $S_1(m,\delta) + S_2(m,\delta)$ is entirely similar. If we leave T_1 and D unspecified but subject to $2 \leqslant D \leqslant A$ and $T_1 \geqslant D$, then, proceeding as in the proof of Lemma 9, we get

$$(8.4)\qquad S_1(m,\delta) + S_2(m,\delta) < h_1(T_1)/\pi + e_1(T_1),$$

where

$$h_1(T) = \frac{2+m\delta}{2}\int_D^T y^{-1}\log\frac{y}{2\pi}\,dy + R_m(\delta)\delta^{-m}\int_T^\infty y^{-m-1}\log\frac{y}{2\pi}\,dy$$

$$+ (2+m\delta)\pi\left\{G(D) + \frac{1}{4\pi}\log^2\frac{D}{2\pi}\right\},$$

$$G(D) = \sum_{0<\gamma\leqslant D}\left(\gamma^2+\frac{1}{4}\right)^{-1/2} - \frac{1}{4\pi}\left\{\left(\log\frac{D}{2\pi}-1\right)^2+1\right\}$$

$$(8.5)\qquad + \frac{1}{D}\left\{0.137\log D + 0.443\left(\log\log D + \frac{1}{\log D}\right) + 2.6 - N(D)\right\},$$

$$e_1(T) = -\frac{2\pi}{T}\left\{\frac{2+m\delta}{2}\left(0.137 + \frac{0.443}{\log D}\right) - \frac{R_m(\delta)}{(m+1)(\delta T)^m}\left(0.137 + \frac{0.443}{\log T}\right)\right\}$$

$$+ \frac{2\pi}{T}\left\{\frac{2+m\delta}{2} - R_m(\delta)(\delta T)^{-m}\right\}\{N(T) - F(T) - R(T)\}.$$

Here, also, $e_1(T_1)$ is small compared with $h_1(T_1)$. Furthermore,

$$h_1'(T) = \frac{2+m\delta}{2}\,T^{-1}\log\frac{T}{2\pi} - R_m(\delta)\delta^{-m}T^{-m-1}\log\frac{T}{2\pi}$$

$$= T^{-1}\log\frac{T}{2\pi}\cdot\left\{\frac{2+m\delta}{2} - R_m(\delta)(\delta T)^{-m}\right\}.$$

Referring to (8.3), we see that for $T > 2\pi$, $h_1(T)$ is minimal for $T = T_0$.

Let us put

$$(8.6) \qquad C(D) = 4\pi\left(0.137 + \frac{0.443}{\log D}\right), \qquad S(D) = \sum_{0<\gamma\leqslant D}\left(\gamma^2 + \frac{1}{4}\right)^{-1/2}.$$

If we apply (8.4) with $T_1 = T_0$ and replace the term $0.443/\log T_1$ appearing in $e_1(T_1)$ by $0.443/\log D$, then we immediately obtain the following generalized version of Lemma 9 where D is no longer specified by (4.3).

LEMMA 9*. *Let T_0 be defined by* (8.1) *and satisfy $T_0 \geqslant D$, where $2 \leqslant D \leqslant A$. Let m be a positive integer and let $\delta > 0$. Then $S_1(m, \delta) + S_2(m, \delta) < \Omega_1^*$ where*

$$(8.7) \qquad \Omega_1^* = \frac{2 + m\delta}{4\pi}\left\{\left(\log\frac{T_0}{2\pi} + \frac{1}{m}\right)^2 + 4\pi G(D) + \frac{1}{m^2} - \frac{mC(D)}{(m+1)T_0}\right\},$$

and $G(D)$, $C(D)$ are defined by (8.5) *and* (8.6).

The early zeros $\rho_n = \frac{1}{2} + i\gamma_n$ of $\zeta(s)$ have been calculated for $1 \leqslant n \leqslant 12{,}556$ to an accuracy better than $2 \cdot 10^{-7}$ by R. Sherman Lehman as stated in his paper [1966]. From these zeros, the following information was calculated; we note, in passing, that $G(D)$ can be shown to have a limit as $D \longrightarrow \infty$.

D	$N(D)$	$S(D) \leqslant$	$4\pi G(D) \leqslant$	$C(D) \geqslant$
7,436.76651	7,192	3.9674 2351	−0.210075	2.34
8,929.80867	8,896	4.1761 6893	−0.211150	2.33
12,030.00896	12,555	4.5275 6275	−0.212544	2.31

(This may be compared with our earlier choice $D = 158.84998$ which yields $N(D) = 57$, $4\pi G(D) \leqslant 0.038207$ and $C(D) \geqslant 2.82$.) This table was extracted from calculations performed by Professors Lehman and de Vogelaere of the University of California at Berkeley to whom we express our indebtedness and thanks. The computations were done in double precision with fifty-six bits or to more than sixteen significant decimal digits, and due allowance was made for the precision with which Lehman's values of γ_n were computed. The values selected for D were slightly below some γ_{n+1} so that $N(D) = n$ and $S(D) = S(\gamma_n)$. The effect of using the above values of D rather than that in (4.3) is to lower the value of Ω_1 in Theorems 4 and 5 thereby obtaining a smaller value of ϵ in (4.1). Of course, in Theorems 4 and 5, Ω_1 is to be replaced by Ω_1^*, and T_1 is to be replaced by T_0.

We also note that (4.12) and (4.13) can be strengthened considerably by integrating by parts $k + 1$ times. If $x > 0$ we get

$$(8.8) \qquad \Gamma(\nu, x) = G_k(\nu, x) + (\nu - 1)(\nu - 2) \cdots (\nu - k - 1)\Gamma(\nu - k - 1, x),$$

where

$$(8.9) \quad G_k(\nu, x) = x^{\nu-1}e^{-x}\left\{1 + \frac{\nu - 1}{x} + \frac{(\nu-1)(\nu-2)}{x^2} + \cdots + \frac{(\nu-1)\cdots(\nu-k)}{x^k}\right\}.$$

For $\nu < 1$, the last term in (8.8) has the same sign as $(-1)^{k+1}$. As a result, if $x > 0$ and $\nu < 1$, then

$$G_{2l-1}(\nu, x) < \Gamma(\nu, x) < G_{2k}(\nu, x) \tag{8.10}$$

for all positive integers l and k. The most advantageous choice of l and k is such that $2l - 1$ and $2k$ are close to $x + \nu - 1$.

It may also be remarked that the proofs of Theorems 4 and 5 are facilitated by the observation that the definitions (3.7)–(3.10) show that the $S_j(m, \delta)$ decrease as x increases. Hence, one first applies Lemma 8 using x and then replaces x by e^b in all $S_j(m, \delta)$; finally, the resulting $S_j(m, \delta)$ are estimated.

Using the various devices mentioned, we have recalculated the table in Section 5 of this paper to get the one given below. We have used Lemma 9* with the first D given in the small table for $b \leqslant 18.7$; the second value of D was used for $19.0 \leqslant b \leqslant 19.5$; and the third value of D was used for $b \geqslant 20$ although this did not produce any decrease in ϵ for $b \geqslant 40$. We have added entries for $b = 25.32843$, $b = 28.78$ and $b = 550\,(100)\,1050\,(200)\,1450\,(100)\,1650, 1950$. For $b \geqslant 1750$ we have used Theorem 5 and (8.10) with $T_2 = T_0$ and $m = 2$ thereby getting smaller values of ϵ. In addition, we have adjusted the old values of ϵ downward by 1 or 2 units for 6 of the values of b satisfying $800 \leqslant b \leqslant 1350$; as a result, the value of ϵ given in the table below probably does not exceed the value stipulated by Theorems 4 and 5 by more than 2 units throughout the table.

If it is not convenient to use Theorems 4 and 5, then the table will give better bounds for $|\psi(x) - x|/x$ than Theorem 11, provided $\log x \leqslant 2000$. Also, the table gives better results than (7.47) if $\log x \leqslant 1900$; and if $\log x \leqslant 1850$, then the table is better than (7.48). For larger values of x and depending on how near $\log x$ is to the next largest entry b in the table, any one of (7.48), (7.47) or Theorem 11 may provide better results than the table.

We can illustrate the degree of precision of the various bounds by the following small table:

$\log x$	ϵ (table)	$\epsilon_0(x)$ (7.1) value	%	$\epsilon_0(x)E(x)L(v_1)$ (7.47) value	%	$\epsilon_0(x)M(v)L(v)$ (7.48) value	%
2000	6.6880(−7)	8.1913(−7)	22.4	7.6998(−7)	15.1	7.5719(−7)	13.2
4000	2.1591(−9)	2.5021(−9)	15.9	2.3507(−9)	8.9	2.3071(−9)	6.8
10000	2.0331(−14)	2.2817(−14)	12.2	2.1575(−14)	6.1	2.1183(−14)	4.2

Thus, the last entry shows that for $\log x = 10{,}000$ the value $\epsilon_0(x)M(v)L(v) = 2.1183 \cdot 10^{-14}$ is 4.2% in excess of the tabulated value of $\epsilon = 2.0331 \cdot 10^{-14}$. The corresponding figures for $\log x = 100{,}000$ are about 6%, 3%, 2%.

9. Applications. Apart from proving the new Theorem 12, we strengthen the results of Section 5. We need the following result which is an improved version of a result communicated to us by Robert Mandl and included here by his kind permission; this result will be applied with $h(x) = x$ and $\alpha = 1$. To Mandl is also due the idea, occurring in Theorem 12, of using numerical information on the gaps between primes. In what follows, p_n is the nth prime so that $p_1 = 2$.

LEMMA 10. *Let $\eta > 0$, let $h(x) > 0$ be continuous and monotone increasing for $x \geqslant \alpha$, and let P', P be consecutive primes such that $\alpha + \eta h(\alpha) \leqslant P' < P$. Let Q be real and suppose that $(p_{n+1} - p_n)/h(p_n) < \eta$ for all n such that $P \leqslant p_n \leqslant Q$. Let $x_0' \geqslant \alpha$ be the unique solution of $x + \eta h(x) = P'$ and $x_0 > x_0'$ be the unique solution of $x + \eta h(x) = P$. Then $x_0 < P$, and the open interval $(x, x + \eta h(x))$ contains a prime for all x such that $x_0 < x \leqslant Q$. If $P' > x_0$, then the interval $(x, x + \eta h(x))$ contains a prime for all x in the wider range $x_0' < x \leqslant Q$; but if $P' \leqslant x \leqslant x_0$, then $(x, x + \eta h(x))$ does not contain a prime.*

Proof. Clearly, $g(x) \equiv x + \eta h(x)$ is continuous and strictly increasing for $x \geqslant \alpha$. As $g(\alpha) \leqslant P' < P$ and $g(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, we see that the equations $g(x) = P'$, P have unique solutions x_0', x_0 satisfying $\alpha \leqslant x_0' < x_0$. Also, $g(P) > P = g(x_0)$ so that $P > x_0$.

First, suppose that $P \leqslant x \leqslant Q$. Let p_n be the largest prime not exceeding x; then $p_{n+1} > x$ and, as $x \geqslant P$, we have $p_n \geqslant P$ as well as $p_n \leqslant Q$. Hence

$$x < p_{n+1} < p_n + \eta h(p_n) = g(p_n) \leqslant g(x)$$

so that $(x, g(x))$ contains the prime p_{n+1}. Second, if $x_0 < x < P$, then

$$x < P = g(x_0) < g(x);$$

hence $(x, g(x))$ contains the prime P for $x \in (x_0, P)$. Therefore, $(x, g(x))$ contains some prime for all $x \in (x_0, Q]$. If $P' > x_0$ and $x_0' < x < P'$, then the same argument shows that $(x, g(x))$ contains the prime P'; as $P' > x_0$, it follows that $(x, g(x))$ contains a prime for all x satisfying $x_0' < x \leqslant Q$. But, if $P' \leqslant x \leqslant x_0$, then

$$P' \leqslant x < g(x) \leqslant g(x_0) = P$$

so that, since P' and P are consecutive primes, the interval $(x, g(x))$ contains no prime.

THEOREM 12. *The open interval $(x, x + x/16597)$ contains a prime for all $x \geqslant$ 2,010,759.9.*

Proof. Let

$$\lambda(x) = \psi(x) - \theta(x) - \theta(x^{1/2}) - \theta(x^{1/3}) = \sum_{k=4}^{\infty} \theta(x^{1/k}).$$

If $x < 2^{42} = e^{29.11218\ldots}$, then $\theta(x^{1/k}) = 0$ for $k \geqslant 42$. Let $\nu_0 = 61^7 = e^{28.77611\ldots}$, $\nu = 3.155 \cdot 10^{12} = e^{28.78000\ldots}$, and $\nu_1 = 1409^4 = e^{29.00254\ldots}$; then $\nu_0 < \nu < \nu_1$. For $\nu_0 \leqslant x \leqslant y < \nu_1$ and $k \geqslant 4$, we have

$$0 \leqslant \theta(y^{1/k}) - \theta(x^{1/k}) = \sum_{x^{1/k} < p \leqslant y^{1/k}} \log p \leqslant \sum_{\nu_0^{1/k} < p < \nu_1^{1/k}} \log p = 0,$$

except for $k = 4, 5, 8, 18$ where, summing over five consecutive primes when $k = 4$,

$$\theta(y^{1/k}) - \theta(x^{1/k}) \leqslant \begin{cases} \log 1361 + \cdots + \log 1399 < 36.136 & \text{if } k = 4 \\ \log 317, \log 37, \log 5 & \text{if } k = 5, 8, 18. \end{cases}$$

Hence, if $\nu_0 \leqslant x \leqslant y < \nu_1$

$$\lambda(y) - \lambda(x) < 36.136 + \log 317 + \log 37 + \log 5 < 47.12. \tag{9.1}$$

For $x \geqslant \nu$, the table below as well as (5.2), the Corollary of Theorem 6, and (5.1) give

$$\theta(x) = \psi(x) - \theta(x^{1/2}) - \theta(x^{1/3}) - \lambda(x)$$

$$\begin{cases} < (1 + 3.01242 \cdot 10^{-5})x - 0.998684x^{1/2} - 0.985x^{1/3} - \lambda(x) \\ > (1 - 3.01242 \cdot 10^{-5})x - 1.001102(x^{1/2} + x^{1/3}) - \lambda(x). \end{cases}$$

Putting $\eta_0 = 6.025179 \cdot 10^{-5}$, we obtain for $\nu \leqslant x < e^{29}$, as a result of (9.1),

$$\begin{aligned} \theta(x + \eta_0 x) - \theta(x) &> (1 - 3.01242 \cdot 10^{-5})(1 + \eta_0)x - (1 + 3.01242 \cdot 10^{-5})x \\ &\quad - \{1.001102(1 + \eta_0)^{1/2} - 0.998684\}x^{1/2} \\ &\quad - \{1.001102(1 + \eta_0)^{1/3} - 0.985\}x^{1/3} - \{\lambda(x + \eta_0 x) - \lambda(x)\} \\ &> \{1.5749 \cdot 10^{-9} - 2.4482 \cdot 10^{-3}x^{-1/2} - 1.6123 \cdot 10^{-2}x^{-2/3} - 48x^{-1}\}x \\ &\equiv f(x)x, \end{aligned}$$

say. Clearly, $f(x)$ is increasing; as we easily check that $f(\nu) > 0$, it follows that $f(x) > 0$ for all $x \geqslant \nu$. Hence, $\theta(x + \eta_0 x) - \theta(x) > 0$ for $\nu \leqslant x < e^{29}$. If $x \geqslant e^{29}$, then [10, (3.36)] gives, similar to the above,

$$\begin{aligned} \theta(x + \eta_0 x) - \theta(x) &> \psi(x + \eta_0 x) - 1.43\sqrt{(1 + \eta_0)x} - \psi(x) \\ &> (1 - 2.8856 \cdot 10^{-5})(1 + \eta_0)x - (1 + 2.8856 \cdot 10^{-5})x - 1.44\sqrt{x} \\ &> (2.53 \cdot 10^{-6}\sqrt{x} - 1.44)\sqrt{x} > 0. \end{aligned}$$

As $\theta(x + \eta_0 x) - \theta(x) > 0$ for all $x \geqslant \nu$, it follows that the half-open interval $(x, x + \eta_0 x]$ contains a prime. Putting $\eta = 6.02518 \cdot 10^{-5} > \eta_0$, we see that $(x, x + \eta x)$ contains a prime for each $x \geqslant \nu$.

Now Brent [3] has shown that $p_{n+1} - p_n \leqslant 652$ for all $p_n \leqslant 2.686 \cdot 10^{12}$; in a private communication, he has informed me that $p_{n+1} - p_n \leqslant 652$ is valid for all $p_n \leqslant \nu$. Let $P = 11{,}622{,}911$; then, for all n such that $P \leqslant p_n \leqslant \nu$, we have

$$(p_{n+1} - p_n)/p_n \leqslant 652/P < 5.61 \cdot 10^{-5} < \eta.$$

Hence, $(x, x + \eta x)$ contains a prime for all $x \geqslant P$ as a result of Lemma 10. The Appel-Rosser table [1961], or the table in Lander and Parkin [8], shows that the largest prime gap up to P does not exceed 154. If $P_1 = 2{,}745{,}209$ and $P_1 \leqslant p_n \leqslant P$, then $(p_{n+1} - p_n)/p_n < 5.61 \cdot 10^{-5}$ so that $(x, x + \eta x)$ contains a prime for all $x \geqslant P_1$. Putting $P_0 = 2{,}010{,}881$, the D. N. Lehmer [9] and Appel-Rosser tables show that $p_{n+1} - p_n \leqslant 112$ for $P_0 \leqslant p_n \leqslant P_1$; for these n, $(p_{n+1} - p_n)/p_n < 5.57 \cdot 10^{-5} < \eta$. Lemma 10 now shows that $(x, x + \eta x)$ contains a prime for all $x > P_0/(1 + \eta)$ and hence for $x \geqslant 2{,}010{,}759.9$. As $\eta < 1/16597$, this completes the proof.

We observe that if $x_1 = 2{,}010{,}759.8$, then $x_1 > 2{,}010{,}733 = p_{149689}$ and $x_1 + x_1/16597 < x_1 + 121.2 = 2{,}010{,}881 = p_{149690}$; thus, $(x_1, x_1 + x_1/16597)$ does not contain a prime. Hence, the stated lower bound 2,010,759.9 for x is essentially best possible. We remark that Mandl has worked out a comprehensive set of alternative versions of Theorem 12 which are valid in wider regions for x but have 16,597 replaced by correspondingly smaller values.

We now indicate how to modify the proofs in Section 5 to obtain stronger inequalities. In place of Theorem 6, there is the following result which, however, yields no improvement of the Corollary to Theorem 6.

THEOREM 6*. *We have*

$$\theta(x) < 1.001\,093x \quad \text{if} \quad 0 < x, \tag{5.1*}$$

$$0.998697x < \theta(x) \quad \text{if} \quad 1{,}155{,}901 \leqslant x, \tag{5.2*}$$

$$\psi(x) - \theta(x) < 1.001\,093\sqrt{x} + 3x^{1/3} \quad \text{if} \quad 0 < x, \tag{5.3*}$$

$$0.998697\sqrt{x} < \psi(x) - \theta(x) \quad \text{if} \quad 121 \leqslant x. \tag{5.4*}$$

Proof. As in Section 5, it suffices to establish (5.1*) and (5.2*). Putting $c =$ 616.78 72256... we note that

$$\psi(x) - \theta(x) - \theta(x^{1/2}) = \sum_{k=3}^{26} \theta(x^{1/k}) = c, \tag{9.2}$$

for all x satisfying $10^8 \leqslant x < 467^3 = e^{18.43898\ldots}$. If $10^8 \leqslant x < 10007^2$ then $\psi(x) - \theta(x) = \theta(9973) + c > 1.04980 \cdot 10^{-4}x$. By examining each of the intervals $[10007^2, 10009^2)$, $[10009^2, 10037^2)$, $[10037^2, 10039^2)$, $[10039^2, e^{18.43})$, we get

$$\psi(x) - \theta(x) > 1.04517 \cdot 10^{-4}x \quad \text{if} \quad 10^8 \leqslant x < e^{18.43}. \tag{9.3}$$

Using $|\psi(x) - x| < 1.19721 \cdot 10^{-3}x$ from the table, we obtain (5.1*) for $10^8 \leqslant x < e^{18.43}$. If $e^{18.43} \leqslant x < e^{18.45}$, we have $\psi(x) - \theta(x) - \theta(\sqrt{x}) \geqslant c$. Using [10, Theorem 10], we get

$$\psi(x) - \theta(x) > 0.98\sqrt{x} + c > 1.02563 \cdot 10^{-4}x$$

and this leads to (5.1*) for $e^{18.43} \leqslant x < e^{18.45}$. If $x \geqslant e^{18.45}$, then [10, Theorem 10] gives

$$\psi(x) - \theta(x) > \theta(\sqrt{x}) + c + \log 467 > 0.98\sqrt{x} + c + \log 467.$$

Applying the table to the intervals $[e^{18.45}, e^{18.5})$ and $[e^{18.5}, e^{18.7})$, yields (5.1*) for $e^{18.45} \leqslant x < e^{18.7}$. For $x \geqslant e^{18.7}$, we use $\theta(x) \leqslant \psi(x)$ and the table to deduce (5.1*). As (5.1*) holds for $0 < x \leqslant 10^8$ by [10, (4.5)], it has now been proved for all $x > 0$.

By the same reasoning as that which established (9.3), we can prove $\psi(x) - \theta(x) < 1.05128 \cdot 10^{-4}x$ for $10^8 \leqslant x < e^{18.43}$. For these x, we then deduce (5.2*). If $e^{18.43} \leqslant x < e^{18.45}$, we have $\psi(x) - \theta(x) - \theta(\sqrt{x}) \leqslant c + \log 467$; we then apply [10, (4.5)] and the table to obtain (5.2*) for the x mentioned. If $x \geqslant e^{18.45}$, then [10, (3.39)] gives

$$\psi(x) - \theta(x) < 1.02x^{1/2} + 3x^{1/3} < 1.14171 \cdot 10^{-4}x;$$

an application of the table again yields (5.2*) which has therefore been established for all $x \geqslant 10^8$. We use [10, (4.6)] for $2{,}370{,}000 \leqslant x < 10^8$ and then the Appel-Rosser table to prove (5.2*) for $x \geqslant 1{,}346{,}533$. The proof of (5.2*) is completed by a tedious use of the Lehmer table which also shows that (5.2*) is false for x just a bit below 1,155,901.

THEOREM 7*. *We have*

$$|\psi(x) - x| < 0.022\,0646\, x/\log x \quad \text{if} \quad 161{,}971 \leqslant x, \tag{5.5a*}$$

$$-\,0.022\,0646 x/\log x < \psi(x) - x \quad \text{if} \quad 89{,}909 \leqslant x, \tag{5.5b*}$$

$$|\theta(x) - x| < 0.023\,9922\, x/\log x \quad \text{if} \quad 758{,}711 \leqslant x, \tag{5.6*}$$

$$\theta(x) - x < 0.020\,1384\, x/\log x \quad \text{if} \quad 1 < x, \tag{5.6a*}$$

$$|\psi(x) - x|,\quad |\theta(x) - x| < 0.007\,7629 x/\log x \quad \text{if} \quad e^{22} \leqslant x. \tag{5.6b*}$$

Proof. We first prove (5.6b*). If $e^{22} \leqslant x < e^{23}$, we apply the table to get

$$|\psi(x) - x| < 2.9941 \cdot 10^{-4} x < 0.00689 x/\log x.$$

By [10, (3 36)], we have

$$0 \leqslant \psi(x) - \theta(x) < 1.43\sqrt{x} < 0.00053 x/\log x$$

so that (5.6b*) follows for $e^{22} \leqslant x < e^{23}$. We continue to use the table for $b = 23$, 35, 400, 550, 650(50)1050, 1150, 1350 and thereby get the result for $e^{22} \leqslant x < e^{1950}$. For $x \geqslant e^{1950}$, we apply Theorem 11 to complete the proof of (5.6b*).

Next, we apply the table for $b = 19$, 19.5 and use (5.6b*) to get

$$|\psi(x) - x| < 0.018\,7514 x/\log x \quad \text{if} \quad e^{19} \leqslant x. \tag{9.4}$$

By applying the table for $b = 18.45$, 18.5, 18.7 and using (9.4), we obtain

$$|\psi(x) - x| < 0.021\,9022 x/\log x \quad \text{if} \quad e^{18.45} \leqslant x. \tag{9.5}$$

Finally, we use the table with $b = 18.42068$ and 18.43, thereby establishing (5.5a*) for $10^8 \leqslant x$.

If $643{,}000 \leqslant x < 10^8$, then [10, (4.12) and (4.5)] yields

$$\psi(x) - x < 0.022\,0646 x/\log x. \tag{9.6}$$

We then use [10, (4.12)] and the Appel-Rosser table to verify (9.6) for $205{,}950 \leqslant x < 10^8$. By using the value for $\theta(205553)$ in this table and considering each of the intervals [205553, 205721), [205721, 205951), we easily see that $\{x - \theta(x)\}/\sqrt{x} > 0.866$ throughout [205553, 205951); from [10, (4.12)] we get (9.6) for $205{,}553 \leqslant x < 10^8$. Additional applications of [10, (4.12)] and the Appel-Rosser table give a verification of (9.6) for $161{,}971 \leqslant x < 10^8$.

We use [10, (4.11) and (4.6)] to obtain (5.5b*) for $332{,}000 \leqslant x < 10^8$. An application of [10, (4.11)] and the Appel-Rosser table gives (5.5b*) for $89{,}909 \leqslant x < 10^8$. This completes the proof of (5.5b*) and, hence, of (5.5a*).

To prove (5.6a*), we first note that $\psi(x) - \theta(x)$ has the constant value $c' = c + \theta(9973)$ if $10^8 \leqslant x < 10007^2$ where c is given by (9.2). On using the table below, as

b	m	δ	ϵ	b	m	δ	ϵ
18.42	2	2.69(-4)	1.1976(-3)	750	7	2.41(-6)	9.6493(-6)
18.43	2	2.68(-4)	1.1930(-3)	800	7	2.28(-6)	9.1328(-6)
18.44	2	2.67(-4)	1.1885(-3)	850	6	2.45(-6)	8.5876(-6)
18.45	2	2.66(-4)	1.1839(-3)	900	6	2.30(-6)	8.0655(-6)
18.5	2	2.61(-4)	1.1615(-3)	950	6	2.16(-6)	7.5754(-6)
18.7	2	2.45(-4)	1.0765(-3)	1000	5	2.35(-6)	7.0481(-6)
19.0	2	2.24(-4)	9.6161(-4)	1050	5	2.18(-6)	6.5529(-6)
19.5	2	1.97(-4)	8.0011(-4)	1100	5	2.03(-6)	6.0923(-6)
20	3	8.47(-5)	6.5759(-4)	1150	4	2.24(-6)	5.6055(-6)
21	3	5.88(-5)	4.4060(-4)	1200	4	2.06(-6)	5.1392(-6)
22	3	4.61(-5)	2.9941(-4)	1250	4	1.88(-6)	4.7118(-6)
23	4	2.11(-5)	2.0171(-4)	1300	3	2.16(-6)	4.3179(-6)
24	5	1.18(-5)	1.3706(-4)	1350	3	1.94(-6)	3.8789(-6)
25	6	7.75(-6)	9.3932(-5)	1400	3	1.74(-6)	3.4850(-6)
26	8	4.69(-6)	6.5552(-5)	1450	3	1.56(-6)	3.1312(-6)
27	9	3.90(-6)	4.7352(-5)	1500	3	1.41(-6)	2.8135(-6)
28	11	3.05(-6)	3.5927(-5)	1550	3	1.26(-6)	2.5283(-6)
29	11	3.02(-6)	2.8856(-5)	1600	2	1.48(-6)	2.2220(-6)
30	12	2.76(-6)	2.4527(-5)	1650	2	1.29(-6)	1.9368(-6)
35	12	2.73(-6)	1.8314(-5)	1700	2	1.13(-6)	1.6887(-6)
40	12	2.72(-6)	1.7748(-5)	1750	2	9.56(-7)	1.4701(-6)
50	12	2.70(-6)	1.7583(-5)	1800	2	7.96(-7)	1.2680(-6)
75	12	2.66(-6)	1.7285(-5)	1850	2	6.64(-7)	1.0861(-6)
100	12	2.61(-6)	1.6993(-5)	1900	2	5.55(-7)	9.2613(-7)
150	12	2.53(-6)	1.6424(-5)	1950	2	4.65(-7)	7.8760(-7)
200	11	2.64(-6)	1.5830(-5)	2000	2	3.90(-7)	6.6880(-7)
250	11	2.54(-6)	1.5257(-5)	2100	2	2.77(-7)	4.8169(-7)
300	10	2.67(-6)	1.4682(-5)	2200	2	1.98(-7)	3.4752(-7)
350	10	2.56(-6)	1.4104(-5)	2300	2	1.42(-7)	2.5169(-7)
400	10	2.46(-6)	1.3548(-5)	2400	2	1.03(-7)	1.8319(-7)
450	9	2.59(-6)	1.2968(-5)	2500	2	7.54(-8)	1.3405(-7)
500	9	2.48(-6)	1.2407(-5)	2700	2	4.09(-8)	7.2946(-8)
550	8	2.63(-6)	1.1854(-5)	3000	2	1.70(-8)	3.0427(-8)
600	8	2.51(-6)	1.1288(-5)	3500	2	4.30(-9)	7.7413(-9)
650	8	2.39(-6)	1.0748(-5)	4000	2	1.19(-9)	2.1591(-9)
700	7	2.55(-6)	1.0196(-5)	10000	2	1.09(-14)	2.0331(-14)

For b = 18.42068, m = 2 and δ = 2.69(-4), one gets ϵ < 1.19721(-3).
For b = 25.32843, m = 7 and δ = 5.90(-6), one gets ϵ < 8.32458(-5).
For b = 28.78, m = 11 and δ = 3.03(-6), one gets ϵ < 3.01242(-5).

well as the Rosser-Walker table, we obtain for these x

$$\theta(x) - x = \psi(x) - x - c' < \left\{1.19721 \cdot 10^{-3} \log x - 1.0512\,7785 \cdot 10^4 \frac{\log x}{x}\right\} \frac{x}{\log x}$$

$$< 0.020\,1212 x/\log x.$$

We continue in this way with the intervals $[10007^2, 10009^2)$, $[10009^2, 10037^2)$,

$[10037^2, 10039^2)$, $[10039^2, e^{18.43})$ thereby proving (5.6a*) for $10^8 \leqslant x < e^{18.43}$. For $x \geqslant e^{18.43}$, we have $\psi(x) - \theta(x) - \theta(\sqrt{x}) \geqslant c$ so that we obtain from [10, (3.37)]

$$\theta(x) - x \leqslant \{\psi(x) - x\} - 0.98\sqrt{x} - c.$$

Using the table for $b = 18.43$, $18.45, 18.5, 18.7$, we obtain (5.6a*) for $10^8 \leqslant x < e^{19}$. Also, if $x \geqslant e^{19}$, then $\theta(x) - x \leqslant \psi(x) - x$ and (9.4) establish (5.6a*) for all $x \geqslant 10^8$. Finally, (5.6a*) holds for $1 < x < 10^8$ by [10, (4.5)].

Proceeding as above, we find that for $10^8 \leqslant x < 10007^2$

$$\theta(x) - x > -\left\{1.19721 \cdot 10^{-3} \log x + c' \frac{\log x}{x}\right\} \frac{x}{\log x} > -0.023\,9900 \frac{x}{\log x},$$

inasmuch as the expression inside the braces is a decreasing function of x in the stipulated range of x. Continuing as above, we obtain

$$\theta(x) - x > -0.023\,9922x/\log x \quad \text{if} \quad 10^8 \leqslant x < e^{18.43}. \tag{9.7}$$

If $e^{18.43} \leqslant x < e^{18.45}$, then $\psi(x) - \theta(x) - \theta(\sqrt{x}) \leqslant c + \log 467$; we apply the table above and [10, (4.5)] to obtain (9.7). If $e^{18.45} \leqslant x < e^{19}$, we use [10, (4.12)] and (9.5) to obtain (9.7). If $e^{19} \leqslant x$, we use [10, (3.36)] and (9.4) to obtain (9.7) which has now been proved for all $x \geqslant 10^8$. We extend the range of x to $x \geqslant$ 1,400,000 by using [10, (4.6)]. Finally, the Appel-Rosser and Lehmer tables are used to extend (9.7) to $x \geqslant 758{,}711$. On using (5.6a*), we see that (5.6*) has been completely proved.

Of these results, only (5.5a*) and (5.5b*) concerning $\psi(x)$ may hold in a wider range for x. In place of the earlier Corollary 2 of Theorem 7, we have the following result which is proved by using the tables of Appel-Rosser, D. N. Lehmer and Rosser-Walker.

COROLLARY 2*. *For $d \leqslant x$, we have*

$$x - x/(c \log x) < \theta(x) \tag{9.8}$$

for each of the following pairs of values of c and d:

c	41	40	39	37	35	29	25
d	758,231	678,407	644,123	486,377	468,577	315,437	302,969

c	23	19	18	15	13	12	11
d	181,889	120,557	89,513	70,877	48,751	40,813	32,353

c	9	8	7	6	5	9/2	4
d	20,873	19,421	11,923	8,623	5,407	3,527	3,301

c	7/2	10/3	3	5/2	7/3	2	9/5
d	2,657	1,973	1,429	809	599	563	347

c	5/3	7/5	9/7	7/6	8/7	1	4/5
d	227	149	101	67	59	41	2

We conclude with the following strengthened and modified form of Theorem 8. Its proof, apart from using $|x - [x]| < 1$, is similar to the proof of Theorem 8 and is therefore omitted. By introducing additional values for b in the range $900 \leqslant b \leqslant 1950$, it might be possible to reduce the listed values of η_2, η_3 and η_4 to 7.4133, 9562.9 and $1.4594 \cdot 10^7$, respectively.

THEOREM 8*. *If* $x > 1$, *then*

$$(5.8^*) \qquad |\theta(x) - x|, \quad |\theta(x) - [x]|, \quad |\psi(x) - x|, \quad |\psi(x) - [x]| \ < \ \eta_k x/\log^k x,$$

where

$$(5.9^*) \qquad \eta_2 = 8.0720, \quad \eta_3 = 10644, \quad \eta_4 = 1.6570 \cdot 10^7.$$

With regard to the bounds (5.6*), (5.8*) and (7.3) for $|\theta(x) - x|$, we note that

$$\frac{0.023\,9922}{\log x} x < \frac{8.0720}{\log^2 x} x < \frac{10644}{\log^3 x} x < \frac{1.6570 \cdot 10^7}{\log^4 x} x < \epsilon_0(x) x$$

if $\log x$ does not exceed 336, 1318, 1556, 1839, respectively.

Note Added in Proof. From Table 1 of Brent [4], the extended version of this in UMT File 4, *Math. Comp.*, v. 29, 1975, p. 331, and the updated version of this in UMT File 21, *Math. Comp.*, v. 30, 1976, p. 379, as well as the more detailed copy in Brent's possession, it is possible to improve a number of the results in the present paper. Thus, one can show that $\theta(x) < x$ for $0 < x \leqslant 10^{11}$; from this one can improve (5.1*) to get $\theta(x) < 1.000\,081x$ for all $x > 0$. Also, (5.6b*) holds for all $x > 1.04 \cdot 10^7$. These and other improvements will be dealt with in a subsequent paper.

Department of Mathematics
State University of New York at Buffalo
Amherst, New York 14226

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