A Necessary Condition for $A$-Stability of Multistep Multiderivative Methods

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Abstract. The region of absolute stability of multistep multiderivative methods is studied in a neighborhood of the origin. This leads to a necessary condition for $A$-stability. For methods where $\rho(\xi)/(-1)$ has no roots of modulus 1 this condition can be checked very easily. For Hermite interpolatory and Adams type methods a necessary condition for $A$-stability is found which depends only on the error order and the number of derivatives used at $(x_{n+k}, y_{n+k})$.

1. Introduction and Results. A multistep method using higher derivatives for solving the initial value problem $y' = f(x, y), y(a) = \eta$ is given by

$$\sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{j=1}^{l} h^j \sum_{i=0}^{k} \beta_{ji} f^{(j)}(x_{n+i}, y_{n+i}) = 0, \quad n = 0, 1, 2, \ldots$$

$\alpha_i, \beta_{ji}$ are real constants, $\alpha_k \neq 0$, $\sum_{i=0}^{k} |\beta_{ji}| \neq 0$, $|\alpha_0| + \sum_{j=1}^{l} |\beta_{j0}| \neq 0$, $x_n = a + nh$, $h > 0$, and

$$f^{(1)}(x, y) = f(x, y);$$

$$f^{(j+1)}(x, y) = \frac{\partial f^{(j)}(x, y)}{\partial x} + f(x, y) \frac{\partial f^{(j)}(x, y)}{\partial y}, \quad j = 1, 2, \ldots, l - 1.$$

It is well known that the method has order $p$ if

$$\rho(e^z) - \sum_{j=1}^{l} z^j \sigma_j(e^z) = \sum_{j=p+1}^{\infty} C_j z^j, \quad C_{p+1} \neq 0,$$

where $\rho(\xi)$ and $\sigma_j(\xi)$ are the polynomials

$$\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i, \quad \sigma_j(\xi) = \sum_{i=0}^{k} \beta_{ji} \xi^i, \quad j = 1, 2, \ldots, l.$$

We shall always assume that the polynomials $\rho$ and $\sigma_j, j = 1, 2, \ldots, l$, have no common factor. The method is convergent if and only if $p \geq 1$ and $\rho(\xi)$ is a simple von Neumann polynomial; that is, if $\xi$ is a root of $\rho(\xi)$, then $|\xi| \leq 1$; and if $|\xi| = 1$, then it is a simple root (see R. Jeltsch [8]).

If the multistep method (1) is applied to the test equation $y' = \mu y, y(0) = 1, \mu$ complex, then (1) is a linear recurrence relation with constant coefficients. The corresponding characteristic equation is...
(3) \[ \rho(\xi) - \sum_{j=1}^{l} z^j \sigma_j(\xi) = 0, \quad z = \mu h. \]

For each \( z \), (3) has \( k \) roots \( \xi_i(z), i = 1, 2, \ldots, k \). The set \( A = \{ z \mid |\xi_i(z)| < 1, i = 1, 2, \ldots, k \} \) is called the region of absolute stability. Let \( \partial A = \overline{A} - A \), where \( \overline{A} \) is the closure of \( A \). A method is called \( A \)-stable if \( A \) contains the whole left-hand plane \( \text{Re} z < 0 \).

In several articles the boundary \( \partial A \) of \( A \) has been plotted in order to determine whether a method is \( A \)-stable or not, see Brown [1], Enright [4], Jeltsch [7]. However, if all growth parameters \( \lambda_j \), given by (4), are positive, then \( \partial A \) will be extremely close to the imaginary axis for \( z \) close to 0. Roundoff errors may defeat the attempt to determine whether \( \partial A \) is in a neighborhood of \( z = 0 \) in \( \mathbb{H}^+ = \{ z \in \mathbb{C} \mid \text{Re} z > 0 \} \) or in \( \mathbb{H}^- = \{ z \in \mathbb{C} \mid \text{Re} z < 0 \} \). Our results fill this gap. In particular, we shall find a necessary condition for \( A \)-stability. It should be noted that a method which violates this condition may still behave numerically almost like an \( A \)-stable method even though it is not \( A \)-stable. In Section 2 this necessary condition for \( A \)-stability is applied to Hermite interpolatory and Adams-type multistep multiderivative methods; and it is found that these cannot be \( A \)-stable if the error order \( p \) is equal to \( 2l_k + 1 \) modulo 4, where

\[
l_k = \begin{cases} 0 & \text{if } \sum_{j=1}^{l} |\beta_{jk}| = 0, \\ t & \text{if } \sum_{j=t+1}^{l} |\beta_{jk}| = 0 \text{ and } \beta_{tk} \neq 0. \end{cases}
\]

The proofs are given in Section 3.

Let \( \xi_j, j = 1, 2, \ldots, s \), be the roots of \( \rho(\xi) \) with modulus 1. Let us introduce the growth parameters

\[
\lambda_j = a_1(\xi_j)/\xi_j \rho'(\xi_j), \quad j = 1, 2, \ldots, s,
\]

and

\[
\mu_j = \frac{1}{\xi_j \rho'(\xi_j)} \left( a_2(\xi_j) + \xi_j \lambda_j a_1'(\xi_j) - \frac{1}{2} \xi_j^2 \lambda_j^2 \rho''(\xi_j) \right), \quad j = 1, 2, \ldots, s.
\]

Furthermore, let the method have order \( p \geq 1 \). Then we define recursively

\[
c_j = \left( C_j - \sum_{i=1}^{j-p-1} c_{j-i} s_i \right)/s_0, \quad j = p + 1, p + 2, \ldots, 2p,
\]

where \( s_0, s_1, \ldots, s_{p-1} \) are given by

\[
\sum_{j=1}^{l} jz^{j-1} a_j(e^z) = \sum_{i=0}^{p-1} s_i z^i + O(z^p).
\]

For the disk \( \{ z \in \mathbb{C} \mid |z| < R \} \) we shall use the symbol \( D(R) \).

**Theorem 1.** Let the multistep method of form (1) be convergent, of order \( p \geq 1 \) and let \( \rho(\xi) \) have \( s \) roots of modulus 1, \( \xi_i, i = 1, 2, \ldots, s \), with \( \xi_1 = 1 \). Let \( \lambda_i \) be real and positive, \( i = 1, 2, \ldots, s \), and define
where $\lambda_j$ and $\mu_j$ are given by (4) and (5), respectively. Assume that one of the conditions (I), (II$_1$)–(II$_4$) holds, where

(I) $\delta < 0$.

(II$_1$) $\delta > 0$, $p$ odd, $c_{p+1}(-1)^{(p+1)/2} > 0$.

(II$_2$) $\delta > 0$, $p$ even, $c_{p+2q}(-1)^{(p/2)+q} > 0$, $c_{p+2q} = 0$, $j = 1, 2, \ldots, q - 1$, for some $q \leq p/2$.

(II$_3$) $\delta > 0$, $p$ odd, $c_{p+1}(-1)^{(p+1)/2} < 0$.

The numbers $c_p$, $j = p + 1, p + 2, \ldots, 2p$, are given by (6). Then there exists a disk $D = D(R)$, $R > 0$, such that $\gamma = \delta A \cap D$ is a continuously differentiable curve which intersects the real axis and the imaginary axis only at $z = 0$. The imaginary axis is tangent to $\gamma$ at $z = 0$. $\gamma$ divides $D$ in two simply connected regions $D^- = A \cap D$ and $D^+ = D - D^-$, see Figure 1. Moreover, each of the conditions (I), (II$_3$), (II$_4$) implies that $D^- \subset H^+$ while each of the conditions (II$_1$), (II$_2$) implies that $D^+ \supset \{0\} \subset H^+$.

**Figure 1. Absolute stability region in a neighborhood of the origin**

(a) if one of the conditions (I), (II$_3$), (II$_4$) holds

(b) if one of the conditions (II$_1$), (II$_2$) holds

**Remarks.** 1. Using (2) and (7), one finds the explicit formulas

\[
C_n = \frac{1}{n!} \sum_{m=0}^{k} \alpha_m m^n - \sum_{j=1}^{\min\{n, i\}} \frac{1}{(n-j)!} \sum_{m=0}^{k} \beta_j m^{n-j},
\]

(8)

and

\[
s_n = \sum_{j=1}^{\min\{n+1, i\}} \frac{j}{(n+1-j)!} \sum_{m=0}^{k} \beta_j m^{n+1-j}, \quad n = 0, 1, 2, \ldots, p - 1.
\]

(9)
Moreover, from (6), (7) and (2) follows

\[(10)\quad c_{p+1} = C_{p+1}/\rho'(1) \neq 0\]

and

\[(11)\quad c_{p+2} = \left( C_{p+2} - \frac{C_{p+1}}{\rho'(1)} \left( o'_1(1) + 2o_2(1) \right) \right) / \rho'(1).\]

2. Let \( s = 1 \). If \( p \) is odd, then Theorem 1 describes \( \partial A \) close to \( z = 0 \) in all cases since \( c_{p+1} \neq 0 \). The methods with \( p \) even and \( c_{p+2j} = 0, j = 1, 2, \ldots, p/2 \), are not covered by Theorem 1. However, there are only a few methods with this property since one has the following result by Griepentrog [6]. There exists no \( k \)-step method of form (1) with \( k \geq 2 \) and \( s = 1 \) for which \( \partial A \) lies exactly on the imaginary axis in a neighborhood of \( z = 0 \). Moreover, a one-step method of form (1) with \( p \geq 1 \) has \( \partial A \) on the imaginary axis in a neighborhood of \( z = 0 \) if and only if \( \beta_j = (-1)^j \beta_{j0}, j = 1, 2, \ldots, l \).

**Theorem 2.** It is necessary for a method to be \( A \)-stable that all growth parameters are real and nonnegative, \( \delta > 0 \) and either (III) or (IV) holds, where \( \delta \) is defined as in Theorem 1 and

- (III) \( p \) odd, \( c_{p+1}(-1)(p+1)/2 > 0 \).
- (IV) \( p \) even, either \( c_{p+2j} = 0, j = 1, 2, \ldots, p/2 \), or \( c_{p+2j}(-1)(p/2)+q > 0, c_{p+2j} = 0, j = 1, 2, \ldots, q - 1 \), for some \( q \leq p/2 \).

**Remark.** This necessary condition for \( A \)-stability is very easy to check for \( s = 1 \). If \( p \) is odd, only \( c_{p+1} \) has to be calculated. If \( p \) is even one finds for most methods that \( c_{p+2} \neq 0 \); and hence, only \( c_{p+2} \) has to be calculated. The following lemma simplifies the problem of determining the sign of \( c_{p+1} \).

**Lemma.** Let the multistep method using higher derivatives be convergent, then

\[ \text{sign} \rho'(1) = \text{sign} \alpha_k. \]

**Proof.** Since the method is convergent, all roots of \( \rho(\xi) \) and \( \rho'(\xi) \) lie in the unit disk and hence the lemma holds.

2. Application to Hermite Interpolatory and Adams Type Methods.

**Definition 1.** A linear multistep method using higher derivatives of the form

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{i=0}^{k} \sum_{j=1}^{l_i} h^i \beta_{ji} f^{(i)}(x_{n+i}, y_{n+i}) = 0 \]

is called Hermite interpolatory if the error order \( p \) is at least \( \Sigma_{i=0}^{k} l_i + k - 1 \).

In Jeltsch [9] the following theorem is proved.

**Theorem 3.** Let a set of nonnegative integers \( l_0, l_1, \ldots, l_k \) with \( \max_{l=0,1,\ldots,k} l_i = l > 0 \) be given. Then there exists a unique Hermite interpolatory multistep method with the given \( l_i, \alpha_k \neq 0 \) and \( \beta_{i,k} \neq 0 \). The error order is \( p = \Sigma_{i=0}^{k} l_i + k - 1 \) and one has

\[ \text{sign} C_{p+1} = (-1)^k \text{sign} \alpha_k. \]

A similar result can be established for Adams-type methods which are defined as follows.
Definition 2. A linear multistep method using higher derivatives is said to be of Adams type if it is of the form

\[ y_{n+k} - y_{n+k-1} - \sum_{i=0}^{k} \sum_{j=1}^{l_i} h^j \beta^{(j)}(x_{n+i}, y_{n+i}) = 0, \]

and its error order \( p \) is at least \( \sum_{i=0}^{k} l_i \).

Theorem 4. Let a set of nonnegative integers \( l_0, l_1, \ldots, l_k \) with \( \max l_i = l > 0 \) be given. Then there exists a unique Adams-type multistep method with the given \( l_i \) and \( \beta_{i,k} \neq 0 \). The error order is \( p = \sum_{i=0}^{k} l_i \) and one has

\[ \text{sign } C_{p+1} = (-1)^{l} \text{sign } \alpha_k. \]

Using Theorems 2, 3, 4 and the Lemma, one then finds immediately the

Theorem 5. A convergent linear multistep method using higher derivatives which is of Adams type or Hermite interpolatory cannot be \( A \)-stable if the error order \( p \) satisfies

\[ p = 2l_k + 1 \mod 4. \]

Example 1. Brown’s methods are interpolatory with

\[ l_0 = l_1 = \cdots = l_{k-1} = 0, \quad l_k = l. \]

Hence, the methods are not \( A \)-stable if \( p = 2l + 1 \mod 4 \). Especially the methods with \( k = 4, l = 2, p = 5; k = 5, l = 3, p = 7 \) and \( k = 6, l = 4, p = 9 \) are not \( A \)-stable. The method with \( p = 10, k = 7 \) and \( l = 4 \) is not covered by Theorem 5. However, A. H. Sipilä has computed \( C_{11} \) and \( C_{12} \) using rational arithmetic and it was found that

\[ c_{12} = (C_{11}/3p'(1))(-4.653007\ldots). \]

Hence, by our Lemma and Theorem 3, one has \( c_{12}(-1)^{p/2+1} < 0 \). Hence, by Theorem 2 this method is not \( A \)-stable. Note that in Brown [1] the plots of \( \partial A \) lead to the wrong conclusion that these methods are \( A \)-stable.

Example 2. Consider the linear one-step methods using higher derivatives which are based on the \((r, l)\) entry of the Padé table of \( \exp(x) \), see Jeltsch [8] or Ehle [3, p. 89]. These methods have order \( p = r + l \) and are interpolatory. It is known, see Ehle [3], that the methods are \( A \)-stable for \( r = l, l-1, l-2 \). From Theorem 5 it follows that the methods are not \( A \)-stable for \( r = l-3 \). This result has been found by Ehle [3].

Example 3. Enright’s second derivative methods are of Adams type with \( l_0 = l_1 = \cdots = l_{k-1} = 1 \) and \( l_k = 2 \), with order \( p = k + 2 \), see Enright [4]. Using the Lemma and Theorems 1 and 4, one finds that for \( k = 3 \mod 4 \) the region of absolute stability behaves at the origin as given in Figure 1a and for \( k = 5 \mod 4 \) as given in Figure 1b.

3. Proof of the Results.

Proof of Theorem 1. The algebraic function \( \xi(z) \) which satisfies (3) has \( k \) branches \( \xi_j(z) \) with \( \xi_j(0) = \xi_j, j = 1, 2, \ldots, k \). Since \( |\xi_j(z)| < 1 \) for \( j = s+1, s+2, \ldots, k \) there exists a \( D(R_1) \), \( R_1 > 0 \) such that \( |\xi_j(z)| < 1 \) for all \( z \in D(R_1) \).
\( j = s + 1, s + 2, \ldots, k \). \( \xi_j(0), j = 1, 2, \ldots, s \), are simple zeros of \( \rho(\zeta) \); and hence, there exists a disk \( D(R_2), 0 < R_2 < R_1 \), such that the branches \( \xi_j(z) \) are analytic in \( D(R_2) \). By the method of undetermined coefficients one finds

\[
\xi_j(z) = \xi_j(0)(1 + \lambda_jz + \mu_jz^2 + O(z^3)), \quad j = 1, 2, \ldots, s;
\]

and hence,

\[
\left. \frac{d\xi_j(z)}{dz} \right|_{z=0} = \xi_j(0)\lambda_j \neq 0, \quad j = 1, 2, \ldots, s.
\]

Hence, there exists an \( R_3, 0 < R_3 < R_2 \), such that the mapping \( \xi_j(z) \): \( z \rightarrow \xi = \xi_j(z) \) is one to one on \( z \in D(R_3) \). Moreover, \( R_3 \) can be chosen so small that the curves \( \gamma^{(j)} = \{z \in D(R_3) \mid |\xi_j(z)| = 1\} \) are continuously differentiable. Clearly, \( \{0\} \in \gamma^{(j)} \) and from (13) it follows that the imaginary axis is tangent to \( \gamma^{(j)} \) at \( z = 0 \). If \( i \neq j \), then either \( \gamma^{(j)} \cap \gamma^{(i)} \) is a finite set or \( \gamma^{(j)} \cap \gamma^{(i)} \) is a continuous curve which contains \( z = 0 \). Hence, there exists \( \tilde{R}, 0 < \tilde{R} < R_3 \), such that either \( \gamma_j \equiv \gamma_j \) and \( \gamma_j \cap \{z \in D(\widetilde{R}) \mid |\xi_j(z)| = 1\} = \{0\} \) or \( \gamma_j \cap \{\tilde{R}, \widetilde{R}\} = \{0\} \) for \( j = 1, 2, \ldots, s \), where \( \gamma_j = D(\widetilde{R}) \cap \gamma^{(j)} \). Each \( \gamma_j \) separates \( D(\widetilde{R}) \) in the two sets \( D_j^- = \{z \in D(\widetilde{R}) \mid |\xi_j(z)| < 1\} \) and \( D_j^+ = \{z \in D(\widetilde{R}) \mid |\xi_j(z)| > 1\} \). Clearly, \( \{\tilde{R}, 0 \} \subset D_j^-, j = 1, 2, \ldots, s \). We distinguish now two cases:

(i) Consider \( \xi_j(z), j = 2, 3, \ldots, s \). With \( z = iy, y \in (\tilde{R}, \widetilde{R}) \), one finds from

\[
|\xi_j(iy)| = |1 - y^2\Re \mu_j + i(\lambda_jy - y^2\Im \mu_j) + O(y^3)|
\]

\[
= \sqrt{1 - y^2(2\Re \mu_j - \lambda_j^2)} + O(y^3).
\]

(ii) Consider \( \xi_1(z) \). It is well known, see, e.g. Gear [5] that \( \xi_1(z) - e^z = O(z^{p+1}) \). Since \( \xi_1(z) \) is analytic at the origin, we can write

\[
\xi_1(z) = e^z \left( 1 - \sum_{j=p+1}^{2p} c_jz^j + O(z^{2p+1}) \right).
\]

If one substitutes (15) in (3) and uses (2), one finds easily that \( c_j, j = p + 1, p + 2, \ldots, 2p \), are determined by (6) and (7). Note that \( c_j \) is a real number. Let \( p \) be odd. Then \( c_{p+1}z^{p+1} = c_{p+1}(-1)^{p+1}/2 \) is real and nonzero. Hence we find for \( z = iy, y \) real,

\[
|\xi_1(iy)| = |e^{iy}| \left| 1 - c_{p+1}i^{p+1}y^{p+1} + O(y^{p+2}) \right|
\]

\[
= \sqrt{1 - 2c_{p+1}(-1)(p+1)/2y^{p+1} + O(y^{p+2})} \quad \text{for } p \text{ odd.}
\]

Let \( p \) be even. Then \( c_{p+2j}z^{p+2j} = c_{p+2j}(-1)^{(p/2)+j} \) is real for \( j = 1, 2, \ldots, p/2 \). Hence, we find for \( z = iy, y \) real

\[
|\xi_1(iy)| = |e^{iy}| \left| 1 - \sum_{j=1}^{p/2} c_{p+2j}i^{p+2j}y^{p+2j} \right|
\]

\[
= \sqrt{1 - 2 \sum_{j=0}^{p/2} c_{p+2j}(-1)^{(p/2)+j}y^{p+2j} + O(y^{2p+1})} \quad \text{for } p \text{ even.}
\]
Assume now that condition (I) holds. Then it follows from (14) that there
exists $R$, $0 < R < R$, such that $|\xi_j(i\nu)| > 1$ for $\nu$ real, $0 < |\nu| < R$ for at least one
$j \in \{2, 3, \ldots, s\}$. Therefore, $D^+ = A \cap D(R) = \bigcap_{j=1}^{s} D_j^+ \cap D(R) \subset H^-$. If (II), (II1), respectively, hold then by (16), (17) and (14) there exists $R$, $0 < R < R$, such that $|\xi_j(i\nu)| < 1$ and $|\xi_j(i\nu)| < 1$, $j = 2, 3, \ldots, s$, for $\nu$ real, $0 < |\nu| < R$. Therefore, $D^+ = \bigcup_{j=1}^{s} D_j^+ \cap D(R)$ satisfies $D^+ = \{0\} \subset H$ since $D_j^+ \cap D(R) = \{0\} \subset H^+$. Similarly, one finds that (II), (II1) imply $D^- \subset H^-$. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\lambda_j = de^{i\phi}$, $d > 0$, $\phi \in (0, 2\pi)$. Clearly,

$$
\psi = \frac{3\pi}{2} - \frac{\phi}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \quad \text{and} \quad \psi + \phi \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right).
$$

Hence, using (12), one finds

$$
|\xi_j(re^{i\psi})| = |1 + rde^{i(\psi + \phi)} + O(r^2)| > 1
$$

for all $r > 0$, $r$ sufficiently small. Therefore, the method is not $A$-stable. Let $\lambda_j \geq 0$, $j = 1, 2, \ldots, s$, and $\delta < 0$. From (14) follows immediately that the method is not $A$-stable. Similarly, using (16) and (17) one finds that (II), (II1) are necessary for $A$-stability. This establishes Theorem 2.

Proof of Theorem 4. In Jeltsch [9] it is shown that to given nonnegative
integers $l_0, l_1, \ldots, l_k$ with max$l_i = l > 0$ there exists a unique Adams-type method
with the given $l_i$, $\beta_i$, and that the error order $p = \Sigma_{i=0}^{k} l_i$. Hence, it remains to show that

$$(18) \quad \text{sign} \, C_{p+1} = (-1)^l \text{sign} \, \alpha_k.$$ 

To show this we construct the method explicitly. Let $P(x)$ be the interpolation poly-
nomial of degree $\Sigma_{i=0}^{k} l_i - 1$ which satisfies

$$
P^{(j-1)}(x_i) = y_n^{(j)} = f^{(j)}(x_{n+i}, y_{n+i}), \quad j = 1, 2, \ldots, l_i, i = 0, 1, 2, \ldots, k.
$$

The multistep method is obtained by setting

$$(19) \quad y_{n+k} - y_{n+k-1} = \int_{x_{n+k-1}}^{x_{n+k}} P(x) \, dx.
$$

To find the error order and $C_{p+1}$ we apply the method given by (19) to a suffi-
ciently smooth function $y(x)$. Clearly,

$$(20) \quad y'(x) - P(x) = f^*(x) \prod_{i=0}^{k} (x - x_{n+i})^{l_i},
$$

where $f^*(x)$ is the generalized divided difference of the function $y'(x)$ on the set

$S = \{x, x_n, x_n, \ldots, x_n, x_{n+1}, x_{n+1}, \ldots, x_{n+1}, x_{n+2}, \ldots, x_{n+k}, x_{n+k}, \ldots, x_{n+k}\}$,

see e.g. Conte and de Boor [2, p. 223]. Hence,
\[
\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x)) \, dx = \int_{x_{n+k-1}}^{x_{n+k}} f^*(x) \prod_{i=0}^{k} (x - x_{n+i})^{-i} \, dx
\]

\[
= f^*(\eta) \int_{x_{n+k-1}}^{x_{n+k}} \prod_{i=0}^{k} (x - x_{n+i})^{-i} \, dx,
\]

since the factor \(\prod_{i=0}^{k} (x - x_{n+i})^{-i}\) does not change sign in the interval \([x_{n+k-1}, x_{n+k}]\);
and hence, the second mean value theorem of the integral calculus can be applied,
\(\eta \in [x_{n+k-1}, x_{n+k}]\). But \(f^*(\eta) = 1/((p + 1)!) y^{(p+1)}(\eta)\), where \(\eta \in [x_n, x_{n+k}]\);
and hence,
\[
\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x)) \, dx = K h^{p+1} y^{(p+1)}(\eta),
\]

where
\[
K = \frac{1}{(p + 1)!} \int_0^1 \prod_{i=0}^{k} (s + k - 1 - i)^{-i} \, ds.
\]

Using (21), it is easy to see that the method given by (19) is of error order \(p\) and that \(C_{p+1} = K\). From (22) follows that \(\text{sign} C_{p+1} = (-1)^k\). The proof of Theorem 4 is
complete since there exists exactly one Adams-type method.

Acknowledgement. Part of this work has been done while the author attended a
research seminar at the University of Victoria. It is a pleasure to thank the participants
of the seminar, especially A. H. Sipilä and R. Skeel, for stimulating discussions. More-
over I would like to thank A. H. Sipilä for computing the \(C_i\)'s for Brown's methods
on the computer of the University of Waterloo.

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