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On the Smoothness of Best L_2 Approximants from Nonlinear Spline Manifolds*

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Abstract. Let S_n^k be the nonlinear spline manifold of order k and with $n - k$ interior variable knots. We prove that all best $L_2[0, 1]$ approximants from S_n^k to a continuous function on $[0, 1]$ are also continuous there. We also prove that there exists a $C^\infty[0, 1]$ function with no $C^2[0, 1]$ best $L_2[0, 1]$ approximants from S_n^k.

1. Introduction. In this paper, we consider the problem of best approximation of functions in the $L_2[0, 1]$ norm $\|\cdot\| = \|\cdot\|_2$ by splines with free knots. Our results will obviously go through to approximation by Chebyshev splines (cf. [5, p. 516]). Let $\mathbf{t} : 0 = t_1 = \cdots = t_k < t_{k+1} \leqslant \cdots \leqslant t_n < t_{n+1} = \cdots = t_{n+k} = 1$ with $t_{j+k} > t_j$ for $j = 1, \ldots, n$, and let $N_{i,k}(\mathbf{t}, \cdot)$ denote the normalized B-splines with the knot sequence $\mathbf{t}$ (cf. [1]). We also denote by S_n^k the space of all splines of order k and with $n - k$ interior knots $t_{k+1}, \ldots, t_n$ on $[0, 1]$. Because of the condition $t_{j+k} > t_j$, a spline in S_n^k has at worst jump discontinuities (when k interior knots coalesce). It is clear (cf. [1]) that any spline s in S_n^k can be written as

$$s(\cdot) = \sum_{i=1}^{n} A_i N_{i,k}(\mathbf{t}, \cdot),$$

where $A_1, \ldots, A_n$ are suitable constants. Let

$$d_2(f, S_n^k) = \inf\{\|f - s\|: s \in S_n^k\}$$

be the distance from a function f to S_n^k in $L_2[0, 1]$, and if $\hat{s} \in S_n^k$ satisfies $\|\hat{s} - f\| = d_2(f, S_n^k)$, we call $\hat{s}$ a best ($L_2[0, 1]$) spline approximant of f from S_n^k. In the next section we will establish the following result:

THEOREM 2.3. *Let f be a continuous function on* $[0, 1]$. *Then all best spline* $L_2[0, 1]$ *approximants to f from S_n^k are also continuous on* $[0, 1]$.

We remark that the above result also holds for $L_p[0, 1]$, $1 < p < \infty$, by essentially using the same proof as the $L_2[0, 1]$ case. Schumaker [7] has proved that every continuous function on $[0, 1]$ has a best (uniform) approximant from S_n^k which is also continuous. Later in [8], he has also proved that if $f \in C^1[0, 1]$ and $n \geqslant 2$, then f has a $C^1[0, 1]$ best (uniform) approximant from S_n^k; but on the other hand if $n \geqslant 3$, there exists a $C^\infty[0, 1]$ function which has no $C^2[0, 1]$ best (uniform) approximants in S_{2n-2}^n. All the above-mentioned results of Schumaker are in the uniform norm. In order to prove an analogous negative result for $L_2[0, 1]$ we need develop some results

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in nonlinear approximation theory in Section 3, and will establish the following theorem in Section 4:

THEOREM 4.1. *There exists a* $C^\infty[0, 1]$ *function* f *which has no* $C^2[0, 1]$ *best* $L_2[0, 1]$ *approximants from* S_n^k, $n \geqslant 2k - 2$.

2. Continuous Best Spline Approximants. In this section, we show that every continuous function on $[0, 1]$ has only continuous best spline approximants in $L_p[0, 1]$, $1 < p < \infty$. In order to establish this result we first prove a theorem concerning best spline approximants from S_n^k with knots of multiplicity k. This theorem which will be Theorem 2.1 below is of independent interest. As a final application of this theorem, a result on discontinuous best spline approximations will be derived.

Let $\mathbf{t}$ be a knot sequence as defined in Section 1 and $s(\mathbf{t}, \cdot)$ be a best approximant to a function f from S_n^k in $L_2[0, 1]$. Let the error be $e(\mathbf{t}, \tau) = f(\tau) - s(\mathbf{t}, \tau)$. In addition, for a fixed m, $1 \leqslant m \leqslant n$, we define $e(\mathbf{t}^-, \tau) = e(\mathbf{t}_m^-, \tau) = \lim_{\epsilon \to 0^+} e(\mathbf{t}_{m,\epsilon}, \tau)$ where $\mathbf{t}_{m,\epsilon} = \mathbf{t} - \epsilon(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the mth component and we define $e(\mathbf{t}^+, \tau)$ analogously. Finally, we set $e(\mathbf{t}, t^-) = \lim_{\epsilon \to 0^+} e(\mathbf{t}, t - \epsilon)$. With this notation we have the following:

THEOREM 2.1. *Let* $s(\mathbf{t}, \cdot)$ *be in* S_n^k, $f \in L_2[0, 1]$, $\|f - s\|_2 = d_2(f, S_n^k)$ *and* $t_m = \cdots = t_{m+k-1}$. *Then if* f *possesses left and right limits at* t_m *we have*

(i) $e(\mathbf{t}, t_m^-)(A_{m-1} - A_m) \geqslant 0$,

(ii) $e(\mathbf{t}, t_m^+)(A_{m-1} - A_m) \leqslant 0$,

where

$$s(\mathbf{t}, \tau) = \sum_{j=1}^{n} A_j N_{j,k}(\mathbf{t}, \tau).$$

Proof. If $s(\mathbf{t}, \cdot)$ satisfies $\|f - s\|_2 = d_2(f, S_n^k)$, then clearly

$$0 \geqslant \left\{\frac{\partial}{\partial t_m} \int_0^1 [e(\mathbf{t}, \tau)]^2 \, d\tau \right\}\Bigg|_{\mathbf{t}^-} \quad \text{and} \quad 0 \leqslant \left\{\frac{\partial}{\partial t_{m+k-1}} \int_0^1 [e(\mathbf{t}, \tau)]^2 \, d\tau \right\}\Bigg|_{\mathbf{t}^+}.$$

Computing the derivatives yields

$$\frac{\partial}{\partial t_m} \int_0^1 [e(\mathbf{t}, \tau)]^2 \, d\tau = -2 \int_0^1 e(\mathbf{t}, \tau) \sum_{j=1}^{n} A_j \frac{\partial}{\partial t_m} N_{j,k}(\mathbf{t}, \tau) \, d\tau.$$

From a result of de Boor we have

$$N_{j,k}(\mathbf{t}, \tau) \equiv [t_{j+1}, \ldots, t_{j+k}](\cdot - \tau)_+^{k-1} - [t_j, \ldots, t_{j+k-1}](\cdot - \tau)_+^{k-1}.$$

Furthermore,

$$\frac{\partial}{\partial t_m}[t_{r+1}, \ldots, t_{r+k}](\cdot - \tau)_+^{k-1}$$

$$= \begin{cases} 0 & \text{if } m < r + 1 \text{ or } m > r + k, \\ [t_{r+1}, \ldots, t_m, t_m, t_{m+1}, \ldots, t_{r+k}](\cdot - \tau)_+^{k-1}, & \text{otherwise}, \end{cases}$$

$$\equiv \phi_{r+1}(\tau).$$

Therefore, $\partial N_{j,k}(\mathbf{t}, \tau)/\partial t_m = \phi_{j+1}(\tau) - \phi_j(\tau)$.

Note that each ϕ_j is a (possibly zero) multiple of a normalized B-spline. Hence, the derivative becomes

$$\frac{\partial}{\partial t_m}\int_0^1 e(\mathbf{t}, \tau)^2\, d\tau = -2\int_0^1 e(\mathbf{t}, \tau)\left\{\sum_{j=1}^n A_j(\phi_{j+1} - \phi_j)\right\} d\tau.$$

Evaluating this derivative at $\mathbf{t}_{m,\epsilon}$, one obtains

$$0 \geqslant \underline{\lim}_{\epsilon\to 0+}\ -2\int_0^1 e(\mathbf{t}_{m,\epsilon}, \tau)\left\{\sum_{j=1}^n A_j(\phi_{j-1} - \phi_j)\right\} d\tau$$

$$= \underline{\lim}_{\epsilon\to 0+}\ -2\int_0^1 e(\mathbf{t}_{m,\epsilon}, \tau)\left\{\sum_{j=1}^n \phi_j(A_{j-1} - A_j)\right\} d\tau,$$

where we set $A_0 \equiv 0$. As $\epsilon \longrightarrow 0+$,

$$\phi_j \equiv \phi_j^\epsilon \longrightarrow \begin{Bmatrix} 0 \quad \text{or} \\ \\ N_{j,k}/(t_{j+k} - t_j) \end{Bmatrix} \quad \text{for } j \neq m;$$

and since $s(\mathbf{t}, \tau)$ is a best approximant to $f(\tau)$ from S_n^k, $\phi_j^0 \equiv N_{j,k}/(t_{j+k} - t_j)$ is orthogonal to the error. But

$$\phi_m(\tau) = [t_m - \epsilon, t_m - \epsilon, \ldots, t_{m+k-1}](\cdot - \tau)_+^{k-1} \equiv \hat{N}_{m,k}(\tau)/\epsilon$$

where $\hat{N}_{m,k}$ is a normalized B-spline determined by the partition $\{t_m - \epsilon, t_m - \epsilon, t_{m+1}, \ldots, t_{m+k-1}\}$ and hence,

$$\int_0^1 \hat{N}_{m,k}(\tau)/\epsilon\, d\tau = \int_{t_m-\epsilon}^{t_m} [t_m - \epsilon, t_m - \epsilon, \ldots, t_{m+k-1}](\cdot - \tau)_+^{k-1} = 1/k$$

via the Peano kernel theorem. Thus, as $\epsilon \longrightarrow 0+$ all the terms are orthogonal to the error but ϕ_m, and the mean value of it is $1/k$. It follows that

$$0 \geqslant \underline{\lim}_{\epsilon\to 0+}\ -2\int_0^1 e(\mathbf{t}_{m,\epsilon}, \tau)\left\{\sum_{j=1}^n (A_{j-1} - A_j)\phi_j\right\} d\tau$$

$$= \frac{-2}{k}\, e(\mathbf{t}, t_m^-)(A_{m-1} - A_m);$$

here we use the left continuity of f.

Similarly, for $\mathbf{t}_{m+k-1,-\epsilon}$

$$0 \leqslant \overline{\lim}_{\epsilon\to 0+}\ -2\int_0^1 e(\mathbf{t}_{m+k-1,-\epsilon}, \tau)\left\{\sum_{j=1}^n (A_{j-1} - A_j)\phi_j\right\} d\tau$$

$$= \frac{-2}{k}\, e(\mathbf{t}, t_{m+k-1}^+)(A_{m-1} - A_m).$$

Thus, multiplying both equations by $-k/2$ yields

$$e(\mathbf{t}, t_m^-)(A_{m-1} - A_m) \geqslant 0, \qquad e(\mathbf{t}, t_m^+)(A_{m-1} - A_m) \leqslant 0.$$

This completes the proof of the theorem.

THEOREM 2.2. *Suppose that f possesses left and right limits at t_m, and that $s(\mathbf{t}, \cdot)$ is a best $L_2[0, 1]$ approximant to f from S_n^k. If s is discontinuous at t_m, then either*

$$[s(t, t_m^-), s(t, t_m^+)] \subset [f(t_m^-), f(t_m^+)]$$

or

$$[s(t, t_m^+), s(t, t_m^-)] \subset [f(t_m^+), f(t_m^-)].$$

Proof. Without loss of generality, we can assume that $s(t, t_m^-) = A_{m-1} < s(t, t_m^+) = A_m$. From (i) of Theorem 2.1, we conclude that $f(t_m^-) - s(t, t_m^-) \leqslant 0$ or $f(t_m^-) \leqslant s(t, t_m^-)$. Similarly, (ii) implies $f(t_m^+) \geqslant s(t, t_m^+)$.

We can use this result to conclude that a continuous function must have a continuous spline best approximant.

THEOREM 2.3. *Let f be a continuous function on* $[0, 1]$. *Then all best spline* $L_2[0, 1]$ *approximants to f from S_n^k are continuous on* $[0, 1]$.

The proof is immediate from Theorem 2.2 since a discontinuity in a best $L_2[0, 1]$ approximant s forces a discontinuity in f.

3. Projections onto Nonlinear Manifolds. The goal of this section is to establish Theorem 3.3 and its corollary. Roughly speaking, these results guarantee that certain elements of well-behaved nonlinear manifolds in a Hilbert space have "many" elements projecting onto them via the metric projection. These results will be used in Section 4 to construct examples of C^∞ functions with no C^2 spline best L_2 approximations.

We also state and indicate proofs of similar results in more general Banach spaces in Theorems 3.1 and 3.2.

We first introduce some notation which will expedite the presentation. The notation is the same as found in [3]. X will denote a normed linear space and A a subset of X. Then we set

$$B(x, r) \equiv \{y \in X: \|x - y\| \leqslant r\},$$

$$\operatorname{dist}(x, A) \equiv \inf\{\|x - a\|: a \in A\}, \tag{3.1}$$

$$P_A(x) \equiv \{a \in A: \|x - a\| = \operatorname{dist}(x, A)\}.$$

The mapping $x \longrightarrow P_A(x)$ is called the *metric projection* from X to subsets of A. For each $x \in X$, the elements of $P_A(x)$ are called the *best approximants to x from A.* A point $a \in A$ is a *local best approximant to x from A* if there is a neighborhood U of a such that $a \in P_{U\cap A}(x)$. If a is the only element of $P_{U\cap A}(x)$ for some neighborhood U of x, then a is called a *strict local best approximant to x*. Throughout we will use θ to denote the zero element of any linear space. If A is a cone with vertex at the origin, then $S(A) \equiv \partial B(\theta, 1) \cap A$.

We will be concerned with approximation from subsets M of X which have the following structure (see Braess [2]).

Definition 3.1. A subset M of X is called a *C^1-representable manifold* (with boundary) if for each $m \in M$ there is a relative neighborhood $U \subset M$ of m satisfying the following three properties:

(i) There is a closed convex body $C \subset R^n$, a relatively open subset V of C, and a homeomorphism g: $V \longrightarrow U$. (If $g^{-1}(m) = \theta$ then g is said to be centered for m.)

(ii) The map g is continuously Fréchet differentiable in V. (The Fréchet derivative of g at $a \in R^n$ is denoted by $g'(a)$.)

(iii) Assuming that g is centered for m, there is a continuous map k from U into $g'(\theta) \cdot (\bigcup_{\alpha>0} \alpha C)$ satisfying $k(m) = \theta$ and

$$\|u - m - k(u)\| = o(\|k(u)\|) \quad \text{as } u \longrightarrow m.$$

We define the tangent cone $\mathrm{TC}(m)$, $m \in M$, to be the set of vectors

$$\mathrm{TC}(m) \equiv \left\{ m + g'(\theta) \cdot \left(\bigcup_{\alpha>0} \alpha C \right) \right\},$$

where g is centered for m. (See [3] for a more lengthy discussion.) The normal cone $N(m)$ at $m \in M$ is defined as

$$N(m) = \{y\colon r(m, y) \perp \mathrm{TC}(m)\},$$

where $r(x, y) \equiv \{\lambda y + (1 - \lambda)x\colon \lambda \geqslant 0\}$ and $r(m, y) \perp \mathrm{TC}(m)$ means that $P_{\mathrm{TC}(m)}(y)$ contains m.

For $m, z \in M$, let $\rho(m, y, z)$ be the radius of the smallest ball centered on $r(m, y)$ which contains m and z in its boundary. (If there is no such ball set $\rho(m, y, z) = \infty$.) The *metric radius of curvature*, $\rho(m)$, is defined to be

$$\rho(m) \equiv \inf_{y \in N(m)} \left[\liminf_{z \to m} \{\rho(m, y, z)\colon z \in M\} \right].$$

The *metric curvature* is naturally defined as $\sigma(m) = 1/\rho(m)$.

The folding of a set A at $a \in A$, denoted by $\mathrm{fld}(a)$, is

$$\mathrm{fld}(a) \equiv \sup \{t_0 \in R^1\colon B(a, t) \cap A \text{ is compact and connected for each } t \leqslant t_0\}.$$

An element $m \in M$ is a critical point of $y \in X$ if $y \in N(m)$. The following lemma was proved by Braess in [2].

LEMMA 3.1. *Each local best approximant to y from M (a C^1-representable manifold) is a critical point.*

Finally, we state a fundamental result due to Braess [2], which generalizes Theorem 3.1 of [6].

THEOREM (NONZERO INDEX THEOREM). *Let M be a C^1-representable manifold and let $y \in X$. Suppose that $A = \{m \in M\colon \alpha \leqslant \|m - y\| \leqslant \beta\}$ is compact. If $m_1 \in A$ is a strict local best approximation to y and $m_2 \in A$ satisfies $\|m_2 - y\| \leqslant \|m_1 - y\|$ and if $B(y, \beta) \cap M$ has a connected component containing m_1 and m_2, then there is a critical point $z \in B(y, \beta) \cap M$ of y which is not a strict local best approximation to y from M.*

Although Braess does not quite state the Nonzero Index Theorem as above, it is easily seen from [2] that the above theorem is true.

We will now state and prove several theorems concerning the local behavior of the normals. In particular, for manifolds with bounded curvature and folding bounded away from zero, the normals from nearby points do not intersect near the manifold. More precisely, we have

THEOREM 3.1. *Let M be a C^1-representable manifold in X. Suppose that $\sigma(m)$ is bounded on compact subsets of M and $\mathrm{fld}(m) > 0$ for all $m \in M$. Then for each $m \in M$ there is an $\epsilon > 0$ so that for all $y \in B(m, \epsilon) \cap N(m)$*

$$P_M(y) = m. \tag{3.2}$$

Before proving this result we remark that this is a generalization of Theorem 3.1 of [3] which states that $\sigma(m) < \infty$ for all $m \in M$ implies that $P_M\colon X\backslash M \longrightarrow M$ is a surjection. The proof of this theorem is essentially the same as that of Theorem 5.1 in [3] and will be omitted. As in the proof of Theorem 5.1, one proceeds by contradiction and finally contradicts the bound of $\sigma(m)$ using the Nonzero Index Theorem.

If M is a C^1-representable manifold for which the kernel of $g'(\theta)$ is trivial then it was shown in [3] that $\mathrm{fld}(g'(\theta)) \neq 0$. Thus, we obtain the analog of Theorem 5.2 of [3] as in the following

THEOREM 3.2. *Suppose M is a C^1-representable manifold such that, for every $m \in M$ there is a centered parameterization g satisfying $g'(\theta)b \neq 0$ for every $b \in S(\bigcup_{\alpha>0}\alpha C)$. If the curvature is bounded on compact subsets of M, then for each $m \in M$ there is an $\epsilon > 0$ so that for all $y \in B(m, \epsilon) \cap N(m)$, $P_M(y) = m$.*

We now specialize to the case where X is a Hilbert space. Further, we will assume that M is a C^2-representable manifold. That is, we will assume that each $m \in M$ has a twice continuous Fréchet differentiable centered parameterization g. In this case we can combine the previous results with those of Chui & Smith [4] to obtain the following

THEOREM 3.3. *Let M be a C^2- representable manifold in a Hilbert space such that for every $m \in M$ there is a centered parameterization g satisfying $g'(\theta)b \neq 0$ for all $b \in S(\bigcup_{\alpha>0}\alpha C)$. Then for each $m \in M$ there is an $\epsilon > 0$ so that for all $y \in B(m, \epsilon) \cap N(m)$, $P_M(y) = m$.*

Notice that the proof of this theorem would be trivial if we knew that $\sigma(m)$ was bounded on compact subsets of M. Theorem 3.2 of [3] shows that indeed $\sigma(m)$ is bounded on compact subsets which yields the result.

As a corollary to this theorem we obtain a result which is essential in the construction of counterexamples in the next section.

COROLLARY 3.4. *Let M be a C^2-representable manifold as in Theorem 3.3. Let D be a dense convex subset of the Hilbert space H. If $\mathrm{TC}(m)$ is an affine variety, then there exists $x \in D$, $x \neq m$, so that $P_M(x) = m$.*

The proof of this corollary begins by noting that $N(m)$ is an affine variety of finite codimension. Thus, $D \cap N(m)$ is dense in $N(m)$. Theorem 3.3 guarantees that each $m \in M$ has a relatively open subset of $N(m)$ (e.g. $N(m) \cap B(m, \epsilon)$) projecting onto m. Since $D \cap N(m)$ is dense in $N(m)$, it is easy to see that there is an $x \in D$, $x \neq m$, $x \in N(m) \cap B(m, \epsilon)$ and this x projects onto m.

4. Spline Manifolds. In this section we discuss the sharpness of Theorem 2.3. In particular, using the results in Section 3 we will show that certain $C^\infty[0, 1]$ functions have a unique L_2 best spline approximant which is in C^1 but not in C^2. These results should be contrasted with those of Schumaker in [8].

THEOREM 4.1. *There exists a $C^\infty[0, 1]$ function f which has no $C^2[0, 1]$ best $L_2[0, 1]$ approximants from S_n^k, $n \geqslant 2k - 2$.*

Proof. Let s be a C^1 spline which is not twice continuously differentiable in $S_n^k\backslash S_{m-1}^k$. Then there is an $\epsilon > 0$ so that $B(s, \epsilon) \cap S_n^k \subset S_n^k\backslash S_{n-1}^k$. We will further assume that

$$s = s(\mathbf{t}^*, \cdot) = \sum_{j=1}^{n} \alpha_j^* N_{j,k}(\mathbf{t}^*, \cdot)$$

and that the map

$$g(\boldsymbol{\alpha}, \mathbf{t}, \cdot) = \sum_{j=1}^{n} \alpha_j N_{j,k}(\mathbf{t}, \cdot)$$

is a C^2 imbedding into $B(s, \epsilon) \cap S_n^k$ for $(\boldsymbol{\alpha}, \mathbf{t})$ in a neighborhood of $(\boldsymbol{\alpha}^*, \mathbf{t}^*)$. It is easy to see that we may choose $(\boldsymbol{\alpha}^*, \mathbf{t}^*)$ so that the kernel of g' is trivial for all $(\boldsymbol{\alpha}, \mathbf{t})$ sufficiently near $(\boldsymbol{\alpha}^*, \mathbf{t}^*)$. Now Corollary 3.4 implies the result.

We conclude this section with several remarks concerning future research. We first note that these results are clearly true for Chebyshev spline functions (cf. [5, p. 516]).

It is not at all clear whether a result similar to Theorem 4.1 in L_p for $p \neq 2$ would be true. This is a difficult problem since it is not possible at present to estimate the curvature of the spline manifold in L_p, $p \neq 2$. In addition, the normals in L_p are no longer subspaces.

Finally, the most important question from the viewpoint of this paper would be to decide whether or not there exist C^∞ functions which project onto $C[0, 1]$ but not $C^1[0, 1]$ spline functions in S_n^k.

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