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Odd Perfect Numbers Not Divisible by 3 Are Divisible by at Least Ten Distinct Primes

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Abstract. Hagis and McDaniel have shown that the largest prime factor of an odd perfect number N is at least 100111, and Pomerance has shown that the second largest prime factor is at least 139. Using these facts together with the method we develop, we show that if $3 \nmid N$, N is divisible by at least ten distinct primes.

1. **Introduction.** A positive integer N is called perfect if $\sigma(N) = 2N$, $\sigma(N)$ being the sum of positive divisors of N. No odd perfect (OP) numbers are known; however, it has been proved that if N is OP and $\omega(N)$ denotes the number of distinct prime factors of N, then $\omega(N) \geqslant 5$ by Sylvester (1888), Dickson (1913) and Kanold (1949); $\omega(N) \geqslant 6$ by Gradstein (1925), Kühnel (1949) and Weber (1951); $\omega(N) \geqslant 7$ by Pomerance (1972, [1]) and Robbins (1972); $\omega(N) \geqslant 8$ by Hagis (1975, [3]); and that if N is OP and $3 \nmid N$, then $\omega(N) \geqslant 8$ by Sylvester (1888), and $\omega(N) \geqslant 9$ by Kanold (1949, [6]). Also, it has been proved that if N is OP, then $N > 10^{200}$ by Buxton and Elmore (1976, [5]), the largest prime factor of $N > 100110$ by Hagis and McDaniel (1975, [4]), and the second largest prime factor of $N \geqslant 139$ by Pomerance (1975, [2]).

In this paper we prove

THEOREM. *If N is OP and $3 \nmid N$, $\omega(N) \geqslant 10$.*

2. **Preliminary results.** Throughout this paper let

$$N = \prod_{i=1}^{r} p_i^{a_i},$$

where $p_1 < p_2 < \cdots < p_r$ are odd primes and $a_1, \ldots, a_r$ are positive integers. We call $p_i^{a_i}$ a component of N and write $V_{p_i}(N)$ for a_i.

Euler proved that if N is OP, then for some j, $p_j \equiv a_j \equiv 1\ (4)$ and for $i \neq j$, $a_i \equiv 0\ (2)$. p_j is called the special prime denoted by Π.

LEMMA 1. *Suppose N is OP, $3 \nmid N$, and p^a is a component of N. If $p \equiv 2\ (3)$, then $p \neq \Pi$, and if $p \equiv 1\ (3)$, then $a \not\equiv 2\ (3)$.*

Proof. If $p \equiv 2\ (3)$ and $p = \Pi$, then $\sigma(p^a) \equiv 0\ (3)$ because a is odd, while if $p \equiv 1\ (3)$ and $a \equiv 2\ (3)$, then $\sigma(p^a) \equiv 0\ (3)$, both of which contradict the fact that $3 \nmid N$. Q.E.D.

From Euler's Theorem and Lemma 1 we have

COROLLARY 1. *Suppose N is OP, $3 \nmid N$, and p^a is a component of N. If $p \equiv 1\ (4)$ and $p \equiv 1\ (3)$, then $a = 1, 4, 6, 9, 10, 12, \ldots$; if $p \equiv 1\ (4)$ and $p \equiv 2\ (3)$,*

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then $a = 2, 4, 6, 8, 10, 12, \ldots$; *if* $p \not\equiv 1$ (4) *and* $p \equiv 1$ (3), *then* $a = 4, 6, 10, 12, \ldots$; *if* $p \not\equiv 1$ (4) *and* $p \equiv 2$ (3), *then* $a = 2, 4, 6, 8, 10, 12, \ldots$.

We are interested in finding $p_i^{a_i}$'s for which

$$\prod_{i=1}^{r} S(p_i^{a_i}) = 2, \quad \text{where } S(p^a) = \sigma(p^a)/p^a.$$

Since the accuracy of the computer is limited, we use the inequality

$$(1) \qquad 0.693147180 < \log 2 = \sum_{i=1}^{r} \log S(p_i^{a_i}) < 0.693147181.$$

With nine-digit figures we have sufficient accuracy, and with log we can easily control computational errors involved.

Suppose N is OP, $3 \nmid N$, and p^a is a component of N. We define

$$a(p) = \text{minimum } \{b | b > 1 \text{ is an allowable power of } p \text{ as determined by Corollary 1 and } p^{b+1} > 10^9\}$$

and

$$L(p^a) = \begin{cases} [10^9 \log S(p^a)]\, 10^{-9} & \text{if } a < a(p), \\ \left\lceil 10^9 \log \dfrac{p}{p-1} \right\rceil 10^{-9} & \text{if } a \geqslant a(p), \end{cases}$$

where [ ] is the greatest integer function.

We note that if p and q are odd primes with $p < q$, then for any positive integers a and b

$$S(q^a) < \frac{q}{q-1} < \frac{p+1}{p} \leqslant S(p^b),$$

and so $L(q^a) \leqslant L(p^b)$.

LEMMA 2. *Suppose*

$$N = \prod_{i=1}^{r} p_i^{a_i}$$

is OP and $3 \nmid N$. *Then*

$$(2) \qquad S_r < \sum_{i=1}^{r} L(p_i^{b_i}) < T_r,$$

where $S_r = 0.693147180 - r\, 10^{-9}$, $T_r = 0.693147181 + r\, 10^{-9}$, $b_i = a_i$ *if* $a_i < a(p_i)$, *and* $b_i = a(p_i)$ *if* $a_i \geqslant a(p_i)$.

Proof. Since N is OP, (1) holds. Suppose p^a is a component of N. If $a < a(p)$, then

$$|\log S(p^a) - L(p^a)| < 10^{-9}.$$

If $a \geqslant a(p)$, then

$$\begin{aligned} 10^{-9} &\geqslant \log \frac{p}{p-1} - L(p^a) > \log S(p^a) - L(p^{a(p)}) \\ &\geqslant \log \frac{p^{a+1}-1}{p^{a+1}-p^a} - \log \frac{p}{p-1} = \log\left(1 - \frac{1}{p^{a+1}}\right) = -\sum_{i=1}^{\infty} \frac{1}{i(p^{a+1})^i} \\ &> -\sum_{i=1}^{\infty} \frac{1}{(p^{a+1})^i} = \frac{-1}{p^{a+1}-1} \geqslant -10^{-9}, \end{aligned}$$

and so

$$|\log S(p^a) - L(p^{a(p)})| < 10^{-9}.$$

Hence,

$$\left|\sum_{i=1}^{r} \log S(p_i^{a_i}) - \sum_{i=1}^{r} L(p_i^{b_i})\right| < r\, 10^{-9}, \tag{3}$$

and (2) follows from (1) and (3). Q.E.D.

We also need the following lemmas, which were proved in [1, pp. 269–271]:

LEMMA 3. *If q is a prime for which $q - 1$ is a power of 2, N is OP, and if p^a is a component of N, then*

$$V_q(\sigma(p^a)) = \begin{cases} V_q(a+1) & \text{if } p \equiv 1\ (q), \\ V_q(p+1) + V_q(a+1) & \text{if } p \equiv -1\ (q) \text{ and } p = \Pi, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 4. *If N is OP, p^a is a component of N, and if q is a prime and $q^b | a+1$, then N is divisible by at least b distinct primes $\equiv 1$ (q) other than p.*

LEMMA 5. *If n is OP, 17^a is a component of N, and if $17^a \nmid \Pi + 1$, then N is divisible by at least two primes $\equiv 1$ (17).*

3. **Proof of the Theorem.** In this section, we shall prove that if $3 \nmid N$ and $\omega(N) = 9$, then N is not OP.

LEMMA 6. *If N is OP, $3 \nmid N$, and if $\omega(N) = 9$, then*

$$p_1 = 5, \quad p_2 = 7, \quad p_3 = 11, \quad p_4 = 13, p_5 \leqslant 19, \quad p_6 \leqslant 23, \quad p_7 \leqslant 53,$$
$$p_8 \geqslant 139 \quad \text{and} \quad p_9 > 100110.$$

Proof. By [4] $p_9 > 100110$, and by [2] $p_8 \geqslant 139$. Others follow from

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{17}{16}\frac{19}{18}\frac{23}{22}\frac{29}{28}\frac{139}{138}\frac{100111}{100110} < 2,$$

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{23}{22}\frac{29}{28}\frac{31}{30}\frac{139}{138}\frac{100111}{100110} < 2,$$

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{29}{28}\frac{31}{30}\frac{139}{138}\frac{100111}{100110} < 2,$$

and

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18}\frac{59}{58}\frac{139}{138}\frac{100111}{100110} < 2. \quad \text{Q.E.D.}$$

LEMMA 7. $p_5 = 17$ *in Lemma* 6.

Proof. Suppose $p_5 = 19$. Then $p_6 = 23$, $p_7 = 29$ and $p_8 = 139$ because

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{19}{18}\frac{23}{22}\frac{31}{30}\frac{139}{138}\frac{100111}{100110} < 2$$

and

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{19}{18}\frac{23}{22}\frac{29}{28}\frac{149}{148}\frac{100111}{100110} < 2.$$

Hence

$$N = 5^{a_1}7^{a_2}11^{a_3}13^{a_4}19^{a_5}23^{a_6}29^{a_7}139^{a_8}p_9^{a_9}.$$

Since

$$\frac{5}{4}\frac{7}{6}S(11^2)\frac{13}{12}\frac{19}{18}\frac{23}{22}\frac{29}{28}\frac{139}{138}\frac{100111}{100110} < 2$$

and

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}S(13^1)\frac{19}{18}\frac{23}{22}\frac{29}{28}\frac{139}{138}\frac{100111}{100110} < 2,$$

$a_3 \neq 2$ and $a_4 \neq 1$. Also, $a_2, a_4, a_5, a_8 \neq 2$ by Corollary 1. Since every odd prime factor of $\sigma(p_i^{a_i})$ is a factor of N, $a_1, a_6, a_7 \neq 2$ and $a_1, a_2 \neq 4$ because $31 | \sigma(5^2)$, $71 | \sigma(5^4)$, $2801 | \sigma(7^4)$, $79 | \sigma(23^2)$ and $67 | \sigma(29^2)$. Hence for $1 \leqslant i \leqslant 2$, $a_i \geqslant 6$ and for $3 \leqslant i \leqslant 8$, $a_i \geqslant 4$. Then N is not OP because

$$S(N) > \prod_{i=1}^{8} S(p_i^{a_i}) > 2. \quad \text{Q.E.D.}$$

LEMMA 8. *$17^{a_5} | \Pi + 1$ and $\Pi > 100110$ in Lemma 6.*

Proof. Suppose $17^{a_5} \nmid \Pi + 1$. Since $p_i \not\equiv \pm 1$ (17) for $1 \leqslant i \leqslant 7$, $p_8 \equiv p_9 \equiv 1$ (17) by Lemma 5. If $17^2 | \sigma(p_j^{a_j})$ for $j = 8$ or 9, then by Lemma 3, $17^2 | a_j + 1$, and by Lemma 4 N would be divisible by at least two primes $\equiv 1$ (17) other than p_j. Hence $17^2 \nmid \sigma(p_j^{a_j})$. Since $17 \nmid \sigma(p_i^{a_i})$ for $1 \leqslant i \leqslant 7$, we conclude that $a_5 = 2$, $17 | \sigma(p_8^{a_8})$ and $17 | \sigma(p_9^{a_9})$. Then $p_8 = \sigma(17^2) = 307$, and for $j = 8, 9$, $a_j = 16$, $p_j \neq \Pi$, $5 \nmid \sigma(p_j^{a_j})$, and so for some $1 \leqslant i \leqslant 7$, $5 | \sigma(p_i^{a_i})$. By Lemma 3 and Corollary 1, $p_i = 11, 31$, or 41, and $\sigma(p_i^4) | \sigma(p_i^{a_i})$ because $5 | a_i + 1$; however, $3221 | \sigma(11^4)$, $17351 | \sigma(31^4)$, $579281 | \sigma(41^4)$, and none of these primes $\equiv 1$ (17). Hence $p_i \neq 11, 31, 41$, a contradiction, and $17^{a_5} | \Pi + 1$.

If $a_5 \geqslant 4$, $\Pi \geqslant 2 \cdot 17^4 - 1 = 167041$, while if $a_5 = 2$, $\Pi = p_9 > 100110$ because $p_8 = 307$. Q.E.D.

LEMMA 9. *If $3 \nmid N$, $\omega(N) = 9$, and if $p_8 > 1000$, N is not OP.*

Proof. Suppose N is OP. Then by Lemma 2

$$S_9 < \sum_{i=1}^{9} L(p_i^{b_i}) < T_9.$$

If $a_i < a(p_i)$, $b_i = a_i$, and so every prime factor of $\sigma(p_i^{b_i})$ is a factor of N except when $p_i = \Pi$. Hence if

$$M = \left(\prod_{i=1}^{7} p_i\right)\left(\prod_{i=1\,;b_i<a(p_i)}^{7} \sigma(p_i^{b_i})\right),$$

we have

(4) $$\omega(M) = 7,$$

(5) $$\omega(M) = 8, \quad \text{or}$$

(6) $$\omega(M) = 9.$$

Suppose (4) holds. Since $p_8 > 1000$ and $p_9 > 100110$,

(7) $$S_9 < \sum_{i=1}^{7} L(p_i^{b_i}) + \log\frac{1009}{1008} + \log\frac{100111}{100110} \quad\text{and}\quad \sum_{i=1}^{7} L(p_i^{b_i}) < T_7.$$

Suppose (5) holds, and let p be the prime factor of M other than p_i, $1 \leqslant i \leqslant 7$. Then

(8)
$$\text{if } 1000 < p < 100110,\ S_9 < \sum_{i=1}^{7} L(p_i^{b_i}) + L(p^b) + \log\frac{100111}{100110},$$

$$\text{if } p > 100110,\ S_9 < \sum_{i=1}^{7} L(p_i^{b_i}) + \log\frac{1009}{1008} + L(p^b), \text{ and}$$

$$\sum_{i=1}^{7} L(p_i^{b_i}) + L(p^b) < T_8,$$

where $b \leqslant a(p)$ is an allowable power of p. Suppose (6) holds. Then the two prime factors of M other than p_i, $1 \leqslant i \leqslant 7$, are p_8 and p_9, and

(9) $$p_8 > 1000, \quad p_9 > 100110 \quad\text{and}\quad S_9 < \sum_{i=1}^{9} L(p_i^{b_i}) < T_9.$$

Computer was used to find $\Pi_{i=1}^{7} p_i^{b_i}$ satisfying

(A) (4) and (7),

(B) (5) and (8), or

(C) (6) and (9),

with the following results:

$$5^{12}7^{10}11^{8}13^{9}17^{8}23^{6}29^{6}, \quad 5^{12}7^{10}11^{8}13^{9}17^{8}23^{6}29^{4},$$

$$5^{12}7^{10}11^{8}13^{9}17^{8}23^{4}29^{6}, \quad 5^{12}7^{10}11^{8}13^{9}17^{6}23^{6}29^{6}$$

$$5^{12}7^{10}11^{8}13^{6}17^{8}23^{6}29^{6}, \quad\text{or}\quad 5^{10}7^{10}11^{8}13^{9}17^{8}23^{6}29^{6}.$$

In every case $p_8 \geqslant 3011$ because

$$S(5^{10}7^{10}11^{8}13^{6}17^{6}23^{4}29^{4}3001^{1}) > 2.$$

Then N is not OP because $p_9 \geqslant \Pi > 17^6 - 1$ and

$$S(N) < \frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{17}{16}\,\frac{23}{22}\,\frac{29}{28}\,\frac{3011}{3010}\,\frac{p_9}{p_9 - 1} < 2. \quad \text{Q.E.D.}$$

LEMMA 10. *If* $3 \nmid N$, $\omega(N) = 9$, *and if* $p_8 < 1000$, *N is not OP.*

Proof. Suppose N is OP. Then by Lemma 2

$$S_9 < \sum_{i=1}^{9} L(p_i^{b_i}) < T_9.$$

If

$$M = \left(\sum_{i=1}^{8} p_i\right)\left(\prod_{i=1;\,b_i<a(p_i)}^{7} \sigma(p_i^{b_i})\right),$$

then

(10) $$\omega(M) = 8, \quad \text{or}$$

(11) $$\omega(M) = 9.$$

Suppose (10) holds. Then

(12) $$p_8 < 1000, \quad S_9 < \sum_{i=1}^{8} L(p_i^{b_i}) + \log \frac{100111}{100110}, \quad \text{and}$$
$$\sum_{i=1}^{8} L(p_i^{b_i}) < T_8.$$

Suppose (11) holds. Then the prime factor of M other than p_i, $1 \leqslant i \leqslant 8$, is p_9, and

(13) $$p_8 < 1000, \quad p_9 > 100110 \quad \text{and} \quad S_9 < \sum_{i=1}^{9} L(p_i^{b_i}) < T_9.$$

Computer was used to find $\Pi_{i=1}^{8} p_i^{b_i}$ satisfying

(A) (10) and (12), or

(B) (11) and (13),

with the following results:

$$5^{12}7^{10}11^{8}13^{9}17^{8}19^{6}47^{6}233^{4}, \quad 5^{12}7^{10}11^{8}13^{9}17^{8}19^{6}47^{6}233^{2},$$
$$5^{12}7^{10}11^{2}13^{9}17^{8}19^{6}43^{6}331^{4}, \quad 5^{2}7^{10}11^{8}13^{9}17^{8}19^{6}31^{6}953^{4},$$
$$5^{2}7^{10}11^{2}13^{9}17^{8}19^{6}31^{6}557^{4}, \quad \text{or} \quad 5^{2}7^{10}11^{2}13^{9}17^{8}19^{6}31^{6}557^{2}.$$

Then N is not OP because in every case $p_9 = \Pi \geqslant 2 \cdot 17^8 - 1$ and $S(N) < 2$. Q.E.D.

Lemmas 9 and 10 prove our theorem.

Computer (PDP 11 at the University of Toledo) program run time for Lemmas 9 and 10 was three minutes.

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