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Certain Pure Cubic Fields With Class-Number One

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Abstract. A description is given of the results of some calculations performed to determine the class number of each of the pure cubic fields $Q(\sqrt[3]{q})$, where q ($\equiv -1 \pmod 3$) is a prime and $q < 35{,}100$. The stability of the percentage of these fields having class-number one is examined.

Let q be any prime such that $q \equiv -1 \pmod 3$. In his review of [1] Shanks [7] noted that for primes $< 10{,}000$, the fraction of the pure cubic fields $Q(\sqrt[3]{q})$ with class-number one tended to remain about 48%. In this note we present some results obtained from evaluating the class number h of $Q(\sqrt[3]{q})$ for each $q < 35{,}100$. The calculations were performed by using the methods of [1, Section 5]. Our main purpose is to study the constancy of this "about 48%", since it remains unknown whether or not infinitely many algebraic number fields have $h = 1$.

In Table 1 we give the values of q, the regulator $R(q)$ of $Q(\sqrt[3]{q})$, and J the length of Voronoi's algorithm period for $\sqrt[3]{q}$, such that

$$R(q) > R(r)$$

for all primes r ($r \equiv -1$ (3)) such that $8429 \leqslant r < q$.

TABLE 1

q	R(q)	J	q	R(q)	J
10037	17941.60487	15972	21839	47361.35191	42122
10067	18150.81288	16318	22469	47942.75017	42716
11621	25661.99636	22908	26417	56816.82041	50385
14897	28630.01878	25280	28517	57091.82492	50671
15527	31541.56340	27991	29063	63398.84106	56707
17669	32388.80366	28517	32213	71481.68242	63674
19391	42811.86808	38337	34607	75693.99813	66931

Since so many of the fields have $h = 1$, the regulators are becoming very large; and consequently, the length of time the computer needs to evaluate them is also greatly increasing. It is because of the very large amount of time that the machine was spending in evaluating the regulators that the calculations were suspended when $q > 35{,}100$.

In Table 2 we give for the 1880 primes $q < 35{,}100$, each class number h that occurs, the frequency $f(h)$ with which this h occurs, the value of $100f(h)/1880$, and the smallest value of q such that h is the class number for $Q(\sqrt[3]{q})$, when this h does not occur in Table 1 of [1].

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TABLE 2

h	f(h)	Percentage	q	h	f(h)	Percentage	q
1	890	47.34		56	2	0.11	
2	486	25.85		64	1	0.05	
4	186	9.89		68	1	0.05	
5	49	2.61		70	2	0.11	
7	39	2.07		71	3	0.16	
8	72	3.83		74	1	0.05	
10	17	0.90		80	1	0.05	
11	10	0.53		92	1	0.05	15131
13	3	0.16		95	1	0.05	15795
14	11	0.59		100	1	0.05	31547
16	28	1.49		104	2	0.11	11549
19	1	0.05		110	1	0.05	17333
20	5	0.27		127	1	0.05	
22	6	0.32		128	1	0.05	
23	1	0.05	33149	136	2	0.11	
25	2	0.11	10181	154	1	0.05	
26	2	0.11		175	1	0.05	
28	12	0.64		181	1	0.05	12251
29	2	0.11	13331	200	1	0.05	12197
31	1	0.05	16553	214	1	0.05	16823
32	4	0.21		230	1	0.05	
34	2	0.11		262	1	0.05	28979
37	3	0.16		284	1	0.05	24137
40	4	0.21		340	1	0.05	18257
41	1	0.05		358	1	0.05	27329
44	4	0.21		389	1	0.05	24023
49	1	0.05		748	1	0.05	17573
50	1	0.05	22259	920	1	0.05	17579
52	2	0.11		1442	1	0.05	32771

Let $n(x)$ be the number of primes q ($q \equiv -1 \pmod 3$) which are less than or equal to x, and let $g(x)$ be the number of those primes such that the class number of $Q(\sqrt[3]{q})$ is one. For $x = 1000, 2000, \ldots, 35000$, we give in Table 3 the value of $100g(x)/n(x)$.

TABLE 3

x	100g(x)/n(x)	x	100g(x)/n(x)	x	100g(x)/n(x)
1000	51.72	13000	47.69	25000	47.37
2000	49.35	14000	47.90	26000	47.42
3000	47.30	15000	48.31	27000	47.56
4000	46.76	16000	48.28	28000	47.50
5000	48.82	17000	48.23	29000	47.60
6000	48.74	18000	47.69	30000	47.70
7000	49.01	19000	48.11	31000	47.50
8000	49.51	20000	47.76	32000	47.68
9000	48.40	21000	47.68	33000	47.33
10000	47.65	22000	47.09	34000	47.46
11000	47.55	23000	47.29	35000	47.31
12000	47.11	24000	47.69		

The class number for each of the first 5000 fields $Q(\sqrt{p})$, where $p \equiv 1 \pmod 4$ has been obtained by Kloss, Newman, and Ordman (see [6]). In [6] Shanks presented a table giving the number of values of p in intervals of 1000 which have a particular class number. He noted that, for each successive group of 1000 of these primes, about 80% of the corresponding fields have class-number one. The apparent steadiness of this ratio prompted him to enquire as to whether it might be a fixed number as p tends to infinity.

The data of Hendy [2] seem to indicate that the ratio of the number of $Q(\sqrt{d})$ (d is any square-free positive integer) with class-number one to the number of such fields with genus one is about 80.5%. Hendy suggests that this ratio is $8/\pi^2$ in the limit. Although the figure of 80% given above is for a somewhat different population than that considered by Hendy, he notes that the distributions of [6] and [2] are similar.

Lakein [3], [4] has obtained class numbers for 10000 quartic fields $K_1 = F_1(\sqrt{\pi})$, where $F_1 = Q(i)$ and $\pi = 1 + 4\alpha$ is a prime in F_1, and for 10000 quartic fields $K_3 = F_3(\sqrt{\pi})$, where $F_3 = Q(\rho)$, ($\rho = \frac{1}{2}(-1 + \sqrt{-3})$), and $\pi = a + b\rho$ is a prime in F_3 such that $a \equiv 1 \pmod 4$, $4 \mid b$, $b > 0$. The distributions he obtains for the number of each type of field having a particular class number are very similar to that of [6]. He also gives in [4] a distribution for $Q(\sqrt{p})$ ($p \equiv 1 \pmod 4$) obtained from previous results of Kuroda (see [5]). Kuroda extended the computations of Kloss, Newman, and Ordman to 100811 values of p. The percentage of these fields with class-number one is 77.65. This seems to indicate that this percentage is slowly decreasing as the number of values of p increases.

On examining Table 3, it appears as if the value of $100g(x)/n(x)$ is tending to stay between 47 and 48; however, with x so limited it is impossible to say whether this trend will continue. The analogous ratios for the real quadratic fields and the special quartic fields mentioned above also seems to be stable within the limits of the tables presented in [6], [3] and [4]. However, we have seen that when more data for the real quadratic fields became available, the corresponding ratio decreased. This suggests that perhaps the value of $100g(x)/n(x)$ might also decrease as x gets large.

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