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Computation of the Regular Continued Fraction for Euler's Constant

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Abstract. We describe a computation of the first 20,000 partial quotients in the regular continued fractions for Euler's constant $\gamma = 0.577 \ldots$ and $\exp(\gamma) = 1.781 \ldots$. A preliminary step was the calculation of γ and $\exp(\gamma)$ to 20,700D. It follows from the continued fractions that, if γ or $\exp(\gamma)$ is of the form P/Q for integers P and Q, then $|Q| > 10^{10000}$.

1. Introduction. The regular continued fraction of a real number x is a (possibly terminating) continued fraction of the form

$$x = q_0 + 1/(q_1 + 1/(q_2 + \cdots)),$$

where the q_i are integers called "partial quotients", and $q_i > 0$ if $i > 0$. We define relatively prime integers P_n and $Q_n > 0$ by

$$P_n/Q_n = q_0 + 1/(q_1 + 1/(q_2 + \cdots + 1/(1 + 1/q_n) \ldots)).$$

If necessary to avoid confusion, we write $q_i(x)$ instead of q_i, etc.

Since it is not known whether Euler's constant $\gamma = 0.577 \ldots$ is rational or irrational, there is considerable interest in computing as many terms as possible in its regular continued fraction. We describe a computation of the partial quotients $q_1(\gamma)$, $q_2(\gamma), \ldots, q_{20000}(\gamma)$, and give various statistics concerning them.

Euler [12] suggested that $G = \exp(\gamma)$ could be a more natural constant than γ. Thus, we also computed $q_1(G), \ldots, q_{20000}(G)$. A preliminary step was the computation of γ and G to 20700 decimal places. These decimal values of γ and G, along with the partial quotients $q_i(\gamma)$ and $q_i(G)$ for $i \leqslant 20000$, have been deposited in the UMT file of this journal.

2. Historical Background. Early computations of γ were performed by Euler, Mascheroni, and others: see Glaisher [13]. Adams [1] computed γ to 263 places, and this result was not improved for 74 years until Wrench [20] extended the computation to 328 places, and then Knuth [16] computed 1271 places. Adams, Wrench, and Knuth used the Euler-Maclaurin summation formula applied to the harmonic series, and Knuth found that the computation of the Bernoulli numbers required in the

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Euler-Maclaurin formula was the most time-consuming part of the calculation.

Sweeney [19] suggested a method which avoided the need for any Bernoulli numbers, and used it to find γ to 3566 places. Sweeney's method is described in Section 3 below. Beyer and Waterman [3] used Sweeney's method to obtain a 7114 place value [4]. However, only the first 4879 places of Beyer and Waterman's value are correct. The error was detected when comparing the continued fraction obtained from their 7114D value with that obtained from a 10488D value computed as described below. Beyer and Waterman have now corrected their result [5].

3. Computation of γ. The method used was suggested by Sweeney [19], and depends on the identity

$$\gamma = S(n) - R(n) - \ln(n), \tag{1}$$

where

$$S(n) = \sum_{k=1}^{\infty} \frac{n^k(-1)^{k-1}}{k!k}, \tag{2}$$

$$R(n) = \int_n^{\infty} \frac{\exp(-u)}{u}\,du \sim \frac{\exp(-n)}{n} \sum_{k=0}^{\infty} k!(-n)^{-k}, \tag{3}$$

and n is a positive integer. Using Stirling's approximation, we have

$$\left| R(n) - \frac{\exp(-n)}{n} \sum_{k=0}^{n-2} k!(-n)^{-k} \right| < 3\exp(-2n) \tag{4}$$

and

$$\left| S(n) - \sum_{k=0}^{\lfloor \alpha n \rfloor} \frac{n^k(-1)^{k-1}}{k!k} \right| < \exp(-2n), \tag{5}$$

where $\alpha = 4.3191\ldots$ is the positive root of $\alpha + 2 = \alpha\ln(\alpha)$. Thus, to obtain γ to d decimal places, we took $n \simeq \frac{1}{2}d \cdot \ln(10)$ and used (1), (4) and (5). Because of cancellation when accumulating the sum in (5), it was necessary to use up to $3d/2$ floating decimal places in the working to obtain d places in $S(n)$. However, about $d/2$ floating decimal places were sufficient when approximating $R(n)$.

The method used by Sweeney [19] and Beyer and Waterman [3]–[5] was simpler but more time-consuming: they took $n \simeq d \cdot \ln(10)$, so $R(n)$ could be neglected entirely, but it was necessary to use about $2d$ floating decimal places to compensate for cancellation.

Our computation of γ was performed using a floating-point multiple-precision package [9] on Univac 1108 and 1100/42 computers. The sums in (4) and (5) were accumulated in the obvious way, so the number of arithmetic operations required was $O(d^2)$. The computation was repeated with several different n and various choices of base (b) in the multiple-precision routines. Details are given in Table 1. The 21014D computation required about 28 hours of computer time.

As noted above, Beyer and Waterman's value [4] is correct to only 4879D. Professor Beyer attributes this to a machine error which occurred during his computation

TABLE 1

Computations of γ

decimal places (d)	n in (1)	base (b)
5074	5851	5000
10488	12083	3400
20800	23955	2360
21014	24204	65536

of ln(2). In such a long computation the probability of a machine error occurring may be quite high. Our first attempt to compute γ to 21014D gave an incorrect result, apparently because of a machine error. (See also [10], [11].) However, the computations summarized in Table 1 gave consistent results, so we are confident that γ is known to at least 20800D.

4. Computation of $\exp(\gamma)$. Using our 20800D value of γ, we computed $\exp(2^{-151}\gamma)$ using the Taylor series for $\exp(x)$, and then

$$G = \exp(\gamma) = (\exp(2^{-151}\gamma))^{2^{151}}. \tag{6}$$

The constant 151 was chosen to approximately minimize the computation time [6], [9].

To verify the result, we read in the computed (decimal) value of G, and computed $\ln(G)$ by the Gauss-Salamin algorithm [2], [7], with $b = 10000$. The computed $\ln(G)$ agreed with γ to 20800D.

5. Computation of Regular Continued Fractions. The method used was similar to that suggested by Lehmer [17] and Wrench and Shanks [21]. Lehmer's idea greatly reduces the number of multiple-precision divisions required, and the computer time used in the continued fraction computations was less than 10 percent of the time used to compute γ and G.

Our program kept track of the loss of significance at each stage of the continued fraction computation, and stopped when no more partial quotients could be guaranteed. Thus, we obtained $q_1(\gamma), \ldots, q_{20136}(\gamma)$ and $q_1(G), \ldots, q_{20187}(G)$ from the 20800D values of γ and G. From a theorem of Lévy [14], [15], [18], we had anticipated obtaining about $20800(6 \cdot \ln(2)\ln(10)/\pi^2) \simeq 20181$ partial quotients.

To verify the results, we read in $q_{20136}(\gamma), \ldots, q_1(\gamma)$ and computed $P_{20136}(\gamma)$ and $Q_{20136}(\gamma)$ using the obvious recurrence relations. A multiple-precision division then gave a value which agreed with γ to 20794D. (The loss of 6D was because the stopping criterion in our continued fraction program was rather conservative.) Similarly for $q_{20187}(G), \ldots, q_1(G)$.

TABLE 2

Distribution of first 20,000 *partial quotients for* γ and G

n	number of $q_i(\gamma) = n$	number of $q_i(G) = n$	expected number
1	8355	8238	8300.7
2	3334	3371	3398.5
3	1869	1896	1862.2
4	1178	1218	1177.9
5	821	827	812.8
6	604	597	594.9
7	461	480	454.4
8	347	363	358.4
9	288	312	290.0
10	247	226	239.5
11–20	1128	1178	1168.3
21–50	787	762	782.0
51–100	279	269	276.0
101–1000	266	234	255.5
>1000	36	29	28.8

6. Statistics. Table 2 gives the distribution of the partial quotients $q_1(x), \ldots, q_{20000}(x)$ for $x = \gamma$ and G. From a well-known theorem of Gauss and Kusmin [15], the frequency of occurrence of a partial quotient n in the regular continued fraction of almost all real numbers x is

$$f_n = \log_2(1 + 1/n) - \log_2(1 + 1/(n + 1)).$$

The last column in Table 2 gives the distribution of quotients expected from the Gauss-Kusmin theorem. A chi-squared test did not show any significant difference (at the 5% level) between the actual and expected distributions.

In Table 3 we list all the "large" quotients found, i.e. all $q_i(x) > 2000$ for $x = \gamma$ or G and $i \leqslant 20000$. The only surprising entry is $q_{4294}(G) = 1568705$.

TABLE 3

Large partial quotients

i	$q_i(\gamma)$	i	$q_i(G)$
528	2076	715	2294
5040	11626	1825	2138
7341	4252	2216	2259
8176	8892	4294	1568705
8770	2135	8120	10012
9255	3725	8501	5314
9424	5542	9022	9794
9534	3055	10616	6329
9550	2180	11572	3227
13095	3292	14782	2128
14821	4133	16224	4679
15346	12156	16790	2975
15847	2714	17944	2489
15901	2039	18667	2569
17025	9776	19617	2529
18267	4455		
19519	4941		
19844	3229		

Let $L_n(x) = \ln(Q_n(x))/n$ and $K_n(x) = (q_1(x) \ldots q_n(x))^{1/n}$. From theorems of Lévy [14], [15], [18] and Khintchine [15],

$$\lim_{n\to\infty} L_n(x) = \frac{\pi^2}{12 \cdot \ln(2)} = 1.186569 \ldots$$

and

$$\lim_{n\to\infty} K_n(x) = \exp\left(\sum_{j=2}^{\infty} f_j \ln(j)\right) = 2.685452 \ldots$$

for almost all x. In Table 4 we give $L_n(\gamma)$, $L_n(G)$, $K_n(\gamma)$ and $K_n(G)$ for various $n \leqslant 20000$.

TABLE 4

Lévy and Khintchine statistics

n	$L_n(\gamma)$	$L_n(G)$	$K_n(\gamma)$	$K_n(G)$
100	1.0910	1.1129	2.3938	2.4935
1000	1.2107	1.1724	2.7591	2.6587
2000	1.2027	1.2024	2.7321	2.7491
5000	1.1741	1.1911	2.6390	2.7060
10000	1.1845	1.1912	2.6771	2.7047
20000	1.1891	1.1851	2.6908	2.6843

7. **Consequences.** Let $x = \gamma$ or G. From Theorem 17 of [15], $|Q_n x - P_n| \leqslant |Qx - P|$ for all integers P and Q with $0 < |Q| \leqslant Q_n$. Using $q_1, \ldots, q_{20000}$, we find $Q_{20000}(\gamma) = 5.6 \ldots \times 10^{10328}$ and $Q_{20000}(G) = 3.3 \ldots \times 10^{10293}$. Hence, we have the following result, which makes it highly unlikely that γ or G is rational.

THEOREM. *If γ or $G = P/Q$ for integers P and Q, then $|Q| > 10^{10000}$.*

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