Computation of the Regular
Continued Fraction for Euler's Constant

By Richard P. Brent

Abstract. We describe a computation of the first 20,000 partial quotients in the regular continued fractions for Euler's constant $\gamma = 0.577 \ldots$ and $\exp(\gamma) = 1.781 \ldots$. A preliminary step was the calculation of $\gamma$ and $\exp(\gamma)$ to 20,700D. It follows from the continued fractions that, if $\gamma$ or $\exp(\gamma)$ is of the form $P/Q$ for integers $P$ and $Q$, then $|Q| > 10^{10000}$.

1. Introduction. The regular continued fraction of a real number $x$ is a (possibly terminating) continued fraction of the form

$$x = q_0 + 1/(q_1 + 1/(q_2 + \cdots)),$$

where the $q_i$ are integers called "partial quotients", and $q_i > 0$ if $i > 0$. We define relatively prime integers $P_n$ and $Q_n > 0$ by

$$P_n/Q_n = q_0 + 1/(q_1 + 1/(q_2 + \cdots + 1/(1 + 1/q_n) \ldots)).$$

If necessary to avoid confusion, we write $q_i(x)$ instead of $q_i$, etc.

Since it is not known whether Euler's constant $\gamma = 0.577 \ldots$ is rational or irrational, there is considerable interest in computing as many terms as possible in its regular continued fraction. We describe a computation of the partial quotients $q_1(\gamma)$, $q_2(\gamma), \ldots, q_{20000}(\gamma)$, and give various statistics concerning them.

Euler [12] suggested that $G = \exp(\gamma)$ could be a more natural constant than $\gamma$. Thus, we also computed $q_1(G), \ldots, q_{20000}(G)$. A preliminary step was the computation of $\gamma$ and $G$ to 20700 decimal places. These decimal values of $\gamma$ and $G$, along with the partial quotients $q_i(\gamma)$ and $q_i(G)$ for $i \leq 20000$, have been deposited in the UMT file of this journal.

2. Historical Background. Early computations of $\gamma$ were performed by Euler, Mascheroni, and others: see Glaisher [13]. Adams [1] computed $\gamma$ to 263 places, and this result was not improved for 74 years until Wrench [20] extended the computation to 328 places, and then Knuth [16] computed 1271 places. Adams, Wrench, and Knuth used the Euler-Mac Laurin summation formula applied to the harmonic series, and Knuth found that the computation of the Bernoulli numbers required in the
Euler-Maclaurin formula was the most time-consuming part of the calculation. Sweeney [19] suggested a method which avoided the need for any Bernoulli numbers, and used it to find \( \gamma \) to 3566 places. Sweeney's method is described in Section 3 below. Beyer and Waterman [3] used Sweeney's method to obtain a 7114 place value [4]. However, only the first 4879 places of Beyer and Waterman's value are correct. The error was detected when comparing the continued fraction obtained from their 7114D value with that obtained from a 10488D value computed as described below. Beyer and Waterman have now corrected their result [5].

3. Computation of \( \gamma \). The method used was suggested by Sweeney [19], and depends on the identity

\[
\gamma = S(n) - R(n) - \ln(n),
\]

where

\[
S(n) = \sum_{k=1}^{\infty} \frac{n^k(-1)^{k-1}}{k!k},
\]

\[
R(n) = \int_{n}^{\infty} \frac{\exp(-u)}{u} du \sim \frac{\exp(-n)}{n} \sum_{k=0}^{\infty} \frac{k!(-n)^{-k}}{n^2},
\]

and \( n \) is a positive integer. Using Stirling's approximation, we have

\[
\left| R(n) - \frac{\exp(-n)}{n} \sum_{k=0}^{n-2} \frac{k!(-n)^{-k}}{n^2} \right| < 3 \exp(-2n)
\]

and

\[
\left| S(n) - \sum_{k=0}^{\infty} \frac{n^k(-1)^{k-1}}{k!k} \right| < \exp(-2n),
\]

where \( \alpha = 4.3191 \ldots \) is the positive root of \( \alpha + 2 = \alpha \ln(\alpha) \). Thus, to obtain \( \gamma \) to \( d \) decimal places, we took \( n \approx \frac{d}{2} \cdot \ln(10) \) and used (1), (4) and (5). Because of cancellation when accumulating the sum in (5), it was necessary to use up to \( 3d/2 \) floating decimal places in the working to obtain \( d \) places in \( S(n) \). However, about \( d/2 \) floating decimal places were sufficient when approximating \( R(n) \).

The method used by Sweeney [19] and Beyer and Waterman [3]–[5] was simpler but more time-consuming: they took \( n \approx d \cdot \ln(10) \), so \( R(n) \) could be neglected entirely, but it was necessary to use about \( 2d \) floating decimal places to compensate for cancellation.

Our computation of \( \gamma \) was performed using a floating-point multiple-precision package [9] on Univac 1108 and 1100/42 computers. The sums in (4) and (5) were accumulated in the obvious way, so the number of arithmetic operations required was \( O(d^2) \). The computation was repeated with several different \( n \) and various choices of base \( (b) \) in the multiple-precision routines. Details are given in Table 1. The 21014D computation required about 28 hours of computer time.

As noted above, Beyer and Waterman's value [4] is correct to only 4879D. Professor Beyer attributes this to a machine error which occurred during his computation.
of \ln(2). In such a long computation the probability of a machine error occurring may be quite high. Our first attempt to compute \( \gamma \) to 21014D gave an incorrect result, apparently because of a machine error. (See also [10], [11].) However, the computations summarized in Table 1 gave consistent results, so we are confident that \( \gamma \) is known to at least 20800D.

4. Computation of \( \exp(\gamma) \). Using our 20800D value of \( \gamma \), we computed \( \exp(2^{-151} \gamma) \) using the Taylor series for \( \exp(x) \), and then

\[
G = \exp(\gamma) = (\exp(2^{-151} \gamma))^{2^{151}}.
\]

The constant 151 was chosen to approximately minimize the computation time [6], [9].

To verify the result, we read in the computed (decimal) value of \( G \), and computed \( \ln(G) \) by the Gauss-Salamin algorithm [2], [7], with \( b = 10000 \). The computed \( \ln(G) \) agreed with \( \gamma \) to 20800D.

5. Computation of Regular Continued Fractions. The method used was similar to that suggested by Lehmer [17] and Wrench and Shanks [21]. Lehmer's idea greatly reduces the number of multiple-precision divisions required, and the computer time used in the continued fraction computations was less than 10 percent of the time used to compute \( \gamma \) and \( G \).

Our program kept track of the loss of significance at each stage of the continued fraction computation, and stopped when no more partial quotients could be guaranteed. Thus, we obtained \( q_1(\gamma), \ldots, q_{20136}(\gamma) \) and \( q_1(G), \ldots, q_{20187}(G) \) from the 20800D values of \( \gamma \) and \( G \). From a theorem of Lévy [14], [15], [18], we had anticipated obtaining about 20800(6 \cdot \ln(2)\ln(10)/\pi^2) \approx 20181 partial quotients.

To verify the results, we read in \( q_{20136}(\gamma), \ldots, q_1(\gamma) \) and computed \( P_{20136}(\gamma) \) and \( Q_{20136}(\gamma) \) using the obvious recurrence relations. A multiple-precision division then gave a value which agreed with \( \gamma \) to 20794D. (The loss of 6D was because the stopping criterion in our continued fraction program was rather conservative.) Similarly for \( q_{20187}(G), \ldots, q_1(G) \).
Table 2

<table>
<thead>
<tr>
<th></th>
<th>number of ( q_x(\gamma) = n )</th>
<th>number of ( q_x(G) = n )</th>
<th>expected number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8355</td>
<td>8238</td>
<td>8300.7</td>
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<tr>
<td>2</td>
<td>3334</td>
<td>3371</td>
<td>3398.5</td>
</tr>
<tr>
<td>3</td>
<td>1869</td>
<td>1896</td>
<td>1862.2</td>
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<td>4</td>
<td>1178</td>
<td>1218</td>
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<td>5</td>
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<td>827</td>
<td>812.8</td>
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<td>597</td>
<td>594.9</td>
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<tr>
<td>7</td>
<td>461</td>
<td>480</td>
<td>454.4</td>
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<td>1128</td>
<td>1178</td>
<td>1168.3</td>
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<tr>
<td>21–50</td>
<td>787</td>
<td>762</td>
<td>782.0</td>
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<tr>
<td>51–100</td>
<td>279</td>
<td>269</td>
<td>276.0</td>
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<tr>
<td>101–1000</td>
<td>266</td>
<td>234</td>
<td>255.5</td>
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<tr>
<td>&gt;1000</td>
<td>36</td>
<td>29</td>
<td>28.8</td>
</tr>
</tbody>
</table>

6. Statistics. Table 2 gives the distribution of the partial quotients \( q_x(\gamma) \), \( \ldots, q_{20000}(x) \) for \( x = \gamma \) and \( G \). From a well-known theorem of Gauss and Kusmin [15], the frequency of occurrence of a partial quotient \( n \) in the regular continued fraction of almost all real numbers \( x \) is

\[
f_n = \log_2(1 + 1/n) - \log_2(1 + 1/(n + 1)).
\]

The last column in Table 2 gives the distribution of quotients expected from the Gauss-Kusmin theorem. A chi-squared test did not show any significant difference (at the 5% level) between the actual and expected distributions.

In Table 3 we list all the “large” quotients found, i.e. all \( q_i(x) > 2000 \) for \( x = \gamma \) or \( G \) and \( i \leq 20000 \). The only surprising entry is \( q_{4294}(G) = 1568705 \).
Let \( L_n(x) = \ln(Q_n(x))/n \) and \( K_n(x) = (q_1(x) \ldots q_n(x))^{1/n} \). From theorems of Lévy [14], [15], [18] and Khintchine [15],

\[
\lim_{n \to \infty} L_n(x) = \frac{\pi^2}{12 \cdot \ln(2)} = 1.186569 \ldots
\]

and

\[
\lim_{n \to \infty} K_n(x) = \exp\left(\sum_{j=2}^{\infty} \frac{f_j \ln(j)}{j}\right) = 2.685452 \ldots
\]

for almost all \( x \). In Table 4 we give \( L_n(\gamma) \), \( L_n(G) \), \( K_n(\gamma) \) and \( K_n(G) \) for various \( n \leq 20000 \).
Table 4

Levy and Khintchine statistics

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L_n(\gamma)$</th>
<th>$L_n(G)$</th>
<th>$K_n(\gamma)$</th>
<th>$K_n(G)$</th>
</tr>
</thead>
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<td>2.4935</td>
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<td>1.1724</td>
<td>2.7591</td>
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<td>1.2027</td>
<td>1.2024</td>
<td>2.7321</td>
<td>2.7491</td>
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<td>1.1741</td>
<td>1.1911</td>
<td>2.6390</td>
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<td>1.1845</td>
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<td>2.6771</td>
<td>2.7047</td>
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<td>20000</td>
<td>1.1891</td>
<td>1.1851</td>
<td>2.6908</td>
<td>2.6843</td>
</tr>
</tbody>
</table>

7. Consequences. Let $x = \gamma$ or $G$. From Theorem 17 of [15], $|Q_n x - P_n| \leq |Q x - P|$ for all integers $P$ and $Q$ with $0 < |Q| \leq Q_n$. Using $q_1, \ldots, q_{20000}$, we find $Q_{20000}(\gamma) = 5.6 \ldots \times 10^{10328}$ and $Q_{20000}(G) = 3.3 \ldots \times 10^{10293}$. Hence, we have the following result, which makes it highly unlikely that $\gamma$ or $G$ is rational.

**Theorem.** If $\gamma$ or $G = P/Q$ for integers $P$ and $Q$, then $|Q| > 10^{10000}$.

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