MATHEMATICS OF COMPUTATION, VOLUME 31, NUMBER 139
JULY 1977, PAGES 752–753

Roots of Two Transcendental Equations Involving Spherical Bessel Functions*

By Robert L. Pexton and Arno D. Steiger

Abstract. Roots of the transcendental equations $j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda)$ and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta}$$

for the spherical Bessel functions of the first and second kind, $j_l(z)$ and $y_l(z)$, have been computed. The ranges for the parameter α, the order l and the root index n are: $\alpha = 0.1(0.1)0.7$, $l = 1(1)15$, $n = 1(1)30$.

To determine the electromagnetic eigenfrequencies of a cavity resonator bounded by two perfectly conducting concentric spheres ($r = \alpha R$ and $r = R$, $0 < \alpha < 1$), it is necessary to solve the transcendental equations

$$j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda), \tag{1}$$

and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta}, \tag{2}$$

for $l = 1, 2, 3, \ldots$. The spherical Bessel functions of the first and the second kind, $j_l(z)$ and $y_l(z)$, are defined in [1, p. 437]. The nth root, $\lambda_{l,n}$, of Eq. (1) is proportional to the nth characteristic frequency of the transverse electric 2^l-pole field. The nth root, $\eta_{l,n}$, of Eq. (2) is proportional to the nth characteristic frequency of the transverse magnetic 2^l-pole field.

By virtue of the relation [1, p. 439, Eq. (10.1.21)]

$$\frac{l+1}{z} f_l(z) + \frac{d}{dz} f_l(z) = f_{l-1}(z) \quad \text{where } f_l(z) = \begin{cases} j_l(z), \\ y_l(z), \end{cases}$$

the transcendental equation (2) becomes

Received September 7, 1976; revised October 25, 1976.

AMS (MOS) subject classifications (1970). Primary 65A05; Secondary 65H05.

Key words and phrases. Roots of transcendental equations, spherical Bessel functions, electromagnetic cavity resonators.

*This work was performed under the auspices of the U. S. Energy Research and Development Administration under contract No. W-7405-Eng-48.

$$(2') \qquad \begin{aligned}&(\alpha\eta j_{l-1}(\alpha\eta) - lj_l(\alpha\eta))(\eta y_{l-1}(\eta) - ly_l(\eta))\\ &\quad = (\alpha\eta y_{l-1}(\alpha\eta) - ly_l(\alpha\eta))(\eta j_{l-1}(\eta) - lj_l(\eta)).\end{aligned}$$

Express Eqs. (1) and (2′) each in the form $F(u) = 0$. For a given order l and a fixed parameter α the function $F(u)$ is then evaluated at a sequence of points u_j, for which $u_j - u_{j-1}$ = constant, until a sign change occurs. In this last interval a root is computed to near machine-word accuracy by using a modified Muller technique [2, p. 51]. The spherical Bessel functions are computed by means of Mechel's recurrence method [3, p. 202]. This straightforward procedure is fast, accurate and reliable.

Tables of numerical values for the roots of the two transcendental equations are to be found in the microfiche supplement of this issue. The ranges for the parameter α, the order l and the root index n are

$$\begin{aligned}\alpha &= 0.1(0.1)0.7,\\ l &= 1(1)15,\\ n &= 1(1)30.\end{aligned}$$

The calculations were performed on a CDC 7600 computer. When extending considerably the ranges for l and n, we encountered no difficulties with our program.

The first six roots of Eq. (1) for $l = 0, 1, 2$ and for $\alpha = 1.2, 1.5, 2.0$ are contained in Table 34 of [4]. Nomograms allowing the approximate determination of a few roots of Eq. (1) are presented in [5].

University of California
Lawrence Livermore Laboratory
Livermore, California 94550

1. M. ABRAMOWITZ & I. A. STEGUN (Editors), *Handbook of Mathematical Functions, With Formulas, Graphs and Mathematical Tables*, 3rd ed., Nat. Bur. Standards, Appl. Math. Ser., no. 55, Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1965. MR **31** #1400.

2. J. M. BLATT, "A stable method of inverse interpolation," *Austral. Comput. J.*, v. 7, 1975, pp. 51–57. MR **52** #7077.

3. Fr. MECHEL, "Improvement in recurrence techniques for the computation of Bessel functions of integral order," *Math. Comp.*, v. 22, 1968, pp. 202–205.

4. E. JAHNKE, F. EMDE & F. LÖSCH, *Tables of Higher Functions*, 6th ed., McGraw-Hill, New York; Teubner, Stuttgart, 1960. MR **22** #5140.

5. L. Z. SALČEV & V. B. POPOV, "Determination of the zeros of a cross-product Bessel function," *Proc. Cambridge Philos. Soc.*, v. 74, 1973, pp. 477–483. MR **48** #2449.