Uniqueness of Padé Approximants
From Series of Orthogonal Polynomials

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Abstract. It is proved that whenever a nonlinear Padé approximant, derived from a
series of orthogonal polynomials, exists, it is unique.

Let \( \phi_r(x) \), \( r = 0, 1, 2, \ldots \), be a set of polynomials which are orthogonal on an
interval \([a, b]\), finite, semi-infinite, or infinite, with weight function \( w(x) \), whose inte-
gral over any subinterval of \([a, b]\) is positive; i.e.,

\[ \int_a^b w(x) \phi_r(x) \phi_s(x) \, dx = 0 \quad \text{if } r \neq s. \]  

Then it is known that \( \phi_r(x) \) is a polynomial of degree exactly \( r \).

Suppose now \( f(x) \) is a function which has a formal expansion of the form

\[ f(x) = \sum_{r=0}^{\infty} a_r \phi_r(x) \]  
on \([a, b]\). The \((m, n)\) Padé approximant to \( f(x) \) is defined to be the rational function

\[ S_{m,n}(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{r=0}^{m} p_r \phi_r(x)}{\sum_{s=0}^{n} q_s \phi_s(x)} \]  
having an expansion in \( \phi_r(x) \), \( r = 0, 1, 2, \ldots \), which agrees with that of \( f(x) \) given in
(2) up to and including the term \( a_{m+n} \phi_{m+n}(x) \). It is assumed that the polynomials
\( P(x) \) and \( Q(x) \) have no common factor, apart from a constant, and that \( Q(x) \) does not
vanish on \([a, b]\). It is worth mentioning that the approximations defined above are
the ones called "nonlinear Padé approximants" in [2].

Theorem 1. If \( g(x) \) is any continuous function on \([a, b]\) such that

\[ \int_a^b w(x) g(x) \phi_r(x) \, dx = 0, \quad r = 0, 1, \ldots, k \]  
then \( g(x) \) either changes sign at least \( k \) times in the interval \([a, b]\) or is identically zero.

The proof of this theorem can be found in [1, p. 110].

As a consequence of Theorem 1, it follows that if \( Q(x) \) is nonzero on \([a, b]\),
then \( q_0 \neq 0 \); hence one can normalize \( Q(x) \) by taking \( q_0 = 1 \).

Theorem 2. If the \((m, n)th\) nonlinear Padé approximant \( P(x)/Q(x) \) to \( f \) exists,
in the sense of (3), and, after dividing out common factors, if \( Q \) is of one sign on
\([a, b]\), then it is unique.

Proof. By the definition of \( S_{m,n}(x) = P(x)/Q(x) \) one has

\[ f(x) - S_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r \phi_r(x). \]
If \( \tilde{S}_{m,n}(x) = \overline{P}(x)/\overline{Q}(x) \) is another \((m, n)\) Padé approximant to (1), then

\[
f(x) - \tilde{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r \phi_r(x).
\]

Subtracting (4) from (5) one obtains

\[
S_{m,n}(x) - \tilde{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} (A_r - A_r) \phi_r(x).
\]

Now since \( S_{m,n}(x) \) and \( \tilde{S}_{m,n}(x) \) are continuous on \([a, b]\) so is \( D(x) = S_{m,n}(x) - \tilde{S}_{m,n}(x) \). Then from (6) it follows that \( D(x) \) satisfies

\[
\int_a^b w(x) D(x) \phi_r(x) \, dx = 0, \quad r = 0, 1, \ldots, m + n.
\]

Hence by Theorem 1, \( D(x) \) either changes sign at least \( m + n + 1 \) times on \((a, b)\), or is identically zero there. But

\[
D(x) = \frac{P(x)}{Q(x)} - \frac{\overline{P}(x)}{\overline{Q}(x)} = \frac{P(x)\overline{Q}(x) - \overline{P}(x)Q(x)}{Q(x)\overline{Q}(x)},
\]

i.e., the numerator of \( D(x) \) is a polynomial of degree at most \( m + n \), therefore, can have at most \( m + n \) zeros on \((a, b)\). Since \( Q(x) \) and \( \overline{Q}(x) \) are nonzero on \([a, b]\), \( D(x) \) changes sign at most \( m + n \) times on \((a, b)\). Therefore, \( D(x) \equiv 0 \); hence \( S_{m,n}(x) = \tilde{S}_{m,n}(x) \). Q.E.D.

So far Padé approximants from Legendre series [2] and Chebyshev series have been considered [3], [4]. As is explained in [2], the determination of the \( q_s \), \( s = 1, 2, \ldots, n \), in general, involves the solution of \( n \) nonlinear equations, the determination of the \( p_r \) being trivial then. However, these \( n \) equations may have several solutions. But, as is mentioned in [2], only one solution with \( Q(x) \neq 0 \) on \([a, b]\) has been found for the examples in [2]. By Theorem 2 there is no other solution, and it is at this point that the result of Theorem 2 becomes important.

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