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A Numerical Conception of Entropy for Quasi-Linear Equations

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Abstract. A family of difference schemes solving the Cauchy problem for quasi-linear equations is studied. This family contains well-known schemes such as the decentered, Lax, Godounov or Lax-Wendroff schemes. Two conditions are given, the first assures the convergence to a weak solution and the second, more restrictive, implies the convergence to the solution in Kružkov's sense, which satisfies an entropy condition that guarantees uniqueness. Some counterexamples are proposed to show the necessity of such conditions.

The purpose of this study is the numerical solution of the Cauchy problem

$$(1) \qquad u_t + f(u)_x = 0 \quad \text{if } (x, t) \in \mathbf{R} \times]0, T[,$$

$$(2) \qquad u(x, 0) = u_0(x) \quad \text{if } x \in \mathbf{R},$$

where $u_0 \in L^\infty(\mathbf{R})$, with locally bounded variation on $\mathbf{R}$, $f \in C^1(\mathbf{R})$, and $T > 0$ are given. Section 1 recalls some theoretical results of existence and mainly of uniqueness for problem (1), (2), more particularly Oleĭnik's and Kružkov's results.

Section 2 is devoted to proofs of convergence for a family of numerical schemes; then Section 3 deals with various applications concerning some well-known numerical schemes (Lax, Godounov, Lax-Wendroff schemes, decentered scheme).

1. Since f is nonlinear, a classical solution u of (1), (2) may offer singularities after some value of t, even when u_0 is very regular. With a more general definition of the solution, we can extend u beyond this value of t. The notion of a weak solution represents one of these generalizations, but does not assure the uniqueness of the extension. These singularities of the solution make needless any hypothesis of regularity on the initial value u_0.

DEFINITION 1. *u is a weak solution of* (1), (2) *when* $u \in L^\infty(\mathbf{R} \times]0, T[)$, *and*

$$(3) \qquad \iint_{\mathbf{R} \times]0,T[} \{\phi_t u + \phi_x f(u)\}\, dx\, dt + \int_{\mathbf{R}} \phi(x, 0) u_0(x)\, dx = 0,$$

for all functions ϕ twice continuously differentiable and with compact support on $\mathbf{R} \times]0, T[$ $(\phi \in C_0^2(\mathbf{R} \times]0, T[))$.

By multiplying (1) by ϕ and integrating by parts, we obtain (3). The existence of a weak solution can be proved by the vanishing viscosity method (parameter $\epsilon > 0$) from a quasi-linear problem of parabolic type

$$(4) \qquad \begin{aligned} &(u_\epsilon)_t + f(u_\epsilon)_x = \epsilon (u_\epsilon)_{xx} \quad \text{if } (x, t) \in \mathbf{R} \times]0, T[, \\ &u_\epsilon(x, 0) = u_0(x) \qquad \text{if } x \in \mathbf{R}. \end{aligned}$$

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In the case of the Burgers equation, i.e. when $f(u) = u^2/2$, the Cauchy problem (1), (2) describes the unidimensional flow of a perfect compressible fluid. The discontinuities of the solution correspond to pure shock waves. The introduction of a second member in (4) is equivalent to a little viscosity, which has a regularizing effect on the flow by changing shocks into regions of strong gradient and small thickness. We obtain the flow of a perfect fluid by making the viscosity vanishing. This parabolic regularization method can also be applied to the general case where $f \in C^1(\mathbf{R})$. From a theoretical point of view, the existence of a weak solution of (1), (2) is shown by a compactness argument in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ on the family $\{u_\epsilon\}_{\epsilon>0}$ of solutions of (4), (2) (see Kružkov [4] and Oleĭnik [9]).

Since f is nonlinear, uniqueness of weak solutions for (1), (2) is not true. As soon as discontinuities appear, we may sometimes build several different weak solutions, satisfying the same problem (1), (2).

In order to select the weak solution, the existence of which is established by the vanishing viscosity method, we have to impose a specific additional condition, the entropy condition, so called because of the previous physical analogy.

If u is a weak solution of (1), (2), piecewise continuously differentiable on $\mathbf{R} \times [0, T[$, and with piecewise regular discontinuity lines, then u satisfies the two following properties:

(i) inside domains bounded by discontinuity lines, u is the solution of (1), (2) in the classical sense;

(ii) each discontinuity line satisfies the Rankine-Hugoniot jump equation, binding its velocity s and the intensity of the shock,

$$s(u_2 - u_1) = f(u_2) - f(u_1), \tag{5}$$

where u_1 and u_2 represent the values of u on each side of discontinuity.

Conversely, if u satisfies (2), (i) and (ii), then u is a weak solution of (1), (2) (see Oleĭnik [9]). This characterization enables us to build examples of weak solutions thus, nonuniqueness can be verified and the entropy condition justified.

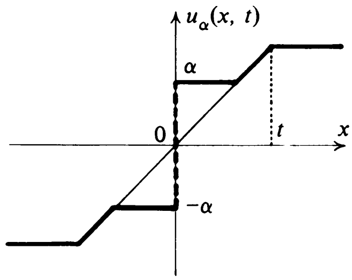
We shall now give three examples of weak solutions.

Example 1. *Burgers Equation with a Rarefaction Wave.* Let $\alpha \in [0, 1]$; the problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x, 0) = sg(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases} \end{cases}$$

admits the weak solution

$$u_\alpha(x, t) = \begin{cases} 1 & \text{if } x \geqslant t, \\ x/t & \text{if } \alpha t \leqslant x \leqslant t, \\ \alpha & \text{if } 0 < x \leqslant \alpha t, \\ -u_\alpha(-x, t) & \text{if } x < 0. \end{cases}$$



The physically right solution corresponds to $\alpha = 0$; it is continuous and shows the dissipation of a rarefaction wave in the time.

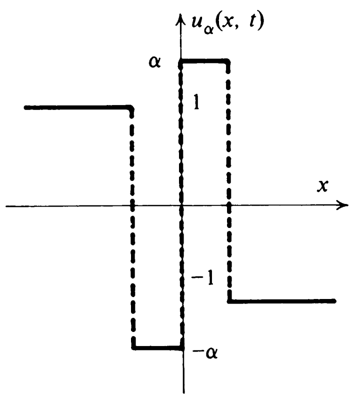
Example 2. *Burgers Equation With a Compression Wave.* Let $\alpha \geqslant 1$; the problem

$$\begin{cases} u_t + uu_x = 0, \\ u(x, 0) = -sg(x), \end{cases}$$

admits the weak solution

$$u_\alpha(x, t) = \begin{cases} 1 & \text{if } x < -\beta t, \\ -\alpha & \text{if } -\beta t < x < 0, \\ \alpha & \text{if } 0 < x < \beta t, \\ -1 & \text{if } \beta t < x, \end{cases}$$

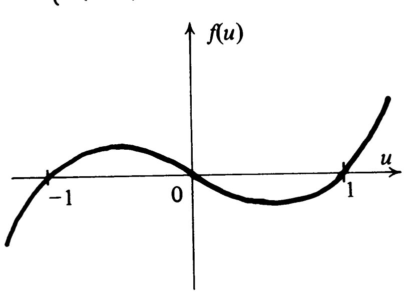
where $\beta = (\alpha - 1)/2$.



The physically right solution corresponds to $\alpha = 1$. In this case, the rarefaction wave is eliminated so we now have a single discontinuity.

Example 3. *A Nonconvex Problem.* The problem

$$\begin{cases} u_t + \tfrac{1}{2}(3u^2 - 1)u_x = 0, \\ u(x, 0) = sg(x), \end{cases}$$



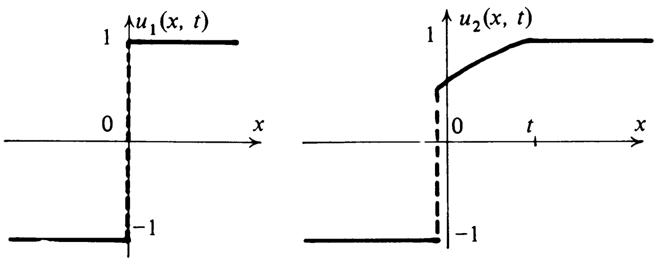
admits the two weak solutions

$$u_1(x, t) = sg(x)$$

and

$$u_2(x, t) = \begin{cases} +1 & \text{if } x > t, \\ \sqrt{\dfrac{2x + t}{3t}} & \text{if } t > x > -t/8, \\ -1 & \text{if } x < -t/8. \end{cases}$$

Now we have $f(u) = u(u^2 - 1)/2$.



Each of the three examples proves the nonuniqueness of weak solutions. In the case of the Burgers equation, eliminating rarefaction waves characterizes the physically right solution. That is the same as avoiding shocks with decreasing entropy. When f is strictly convex, this entropy condition may be written

$$(6) \qquad \forall (x, t) \in \mathbf{R} \times]0, T[, \qquad u(x - 0, t) \geqslant u(x + 0, t).$$

In [10], Oleĭnik proposes a generalization of this entropy condition when f is not assumed to be convex, and then obtains uniqueness. If we put $u_+ = u(x + 0, t)$, $u_- = u(x - 0, t)$, at a point (x, t) of a discontinuity line of the weak solution u, the entropy condition can be written

$$(7) \qquad \forall k \in]\operatorname{Inf}(u_-, u_+), \operatorname{Sup}(u_-, u_+)[, \qquad \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geqslant \frac{f(u_+) - f(k)}{u_+ - k},$$

which is equivalent to

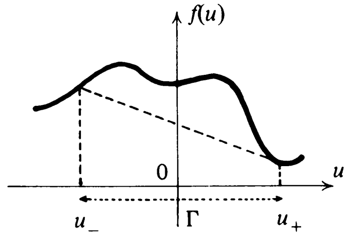
$$(8) \qquad \forall k \in]\operatorname{Inf}(u_-, u_+), \operatorname{Sup}(u_-, u_+)[, \qquad \frac{f(u_+) - f(u_-)}{u_+ - u_-} \leqslant \frac{f(u_-) - f(k)}{u_- - k}.$$

In Example 3, the weak solution u_2 satisfies this entropy condition.

Geometrically, the entropy condition indicates on which side of the straight line from $(u_-, f(u_-))$ to $(u_+ f(u_+))$ the graph of f must be entirely located on the interval bounded by u_- and u_+ and denoted by $\Gamma = [\operatorname{Inf}(u_-, u_+), \operatorname{Sup}(u_-, u_+)]$. If s is the shock velocity, then we deduce from (5), (7) and (8)

$$(9) \qquad |s| = \sup_{k \in \Gamma} \left[\operatorname{Max} \left\{ \frac{f(u_+) - f(k)}{u_+ - k}, -\frac{f(u_-) - f(k)}{u_- - k} \right\} \right].$$

Equation (9) gives the module of the physically right shock velocity in relation to its intensity. For a given intensity, the entropy condition selects the shock for which the velocity module is maximal.



Hopf [3] and Kružkov [4] have proposed a definition of weak solution implicitly containing the entropy condition by starting from (4) rather than from (1), which will now be briefly recalled. Let $h \in C^1(\mathbf{R})$ be nondecreasing. We build two differentiable functions I and F such that $I'(u) = h(u)$, $F'(u) = h(u)f'(u)$. By multiplying (4) by $h(u)$, we get

$$I(u_\epsilon)_t + f(u_\epsilon)_x = I(u_\epsilon)_{xx} - \epsilon h'(u_\epsilon)[(u_\epsilon)_x]^2.$$

Let $\phi \in C_0^2(\mathbf{R} \times]0, t[)$ be nonnegative. Multiplying by ϕ and integrating by parts on $\mathbf{R} \times]0, T[$ gives

$$\text{(10)}\qquad \begin{aligned} &\iint_{\mathbf{R}\times]0,T[} \{\phi_t I(u_\epsilon) + \phi_x F(u_\epsilon)\}\, dx\, dt \\ &\quad = \epsilon \iint_{\mathbf{R}\times]0,T[} \phi_{xx} I(u_\epsilon)\, dx\, dt + \epsilon \iint_{\mathbf{R}\times]0,T[} h'(u_\epsilon) u_{\epsilon x}^2 \phi\, dx\, dt. \end{aligned}$$

The family $\{u_\epsilon\}_{\epsilon>0}$ which is compact in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ (see [4]), contains a convergent sequence towards $u \in L^\infty(\mathbf{R} \times]0, T[)$. Since the last term of (10) is nonnegative, we get at the limit

$$\text{(11)}\qquad \iint_{\mathbf{R}\times]0,T[} (\phi_t I(u) + \phi_x F(u))\, dx\, dt \geqslant 0.$$

With proper choices of h, the Rankine-Hugoniot equation (5) and Oleĭnik's condition of entropy (7) can be deduced from (11) (see [3]). Now, by means of a density argument, (11) holds for all nondecreasing functions.

Let $k \in \mathbf{R}$; by taking $h(u) = sg(u - k)$, we obtain Kružkov's formulation

$$\text{(12)}\qquad \iint_{\mathbf{R}\times]0,T[} [|u - k|\phi_t + sg(u - k)(f(u) - f(k))\phi_x]\, dx\, dt \geqslant 0,$$

from which we can still deduce (5) and (7), with proper choices of k.

Now, we only have to specify the initial value of u, in order to define the solution in Kružkov's sense.

DEFINITION 2. *A solution of* (1), (2) *in Kružkov's sense is a function* $u \in L^\infty(\mathbf{R} \times]0, T[)$ *verifying*:

(i) (12) *for all* $k \in \mathbf{R}$ *and all functions* $\phi \in C_0^2(\mathbf{R} \times]0, T[)$ *nonnegative,*

(ii) *for all* $R > 0$, *for a negligible set* $E \subset]0, T[$

$$\lim_{t\to 0;t\notin E} \int_{|x|<R} |u(x, t) - u_0(x)|\, dx = 0. \tag{13}$$

In [4], Kruzkov proves existence and uniqueness of such a solution with only the hypotheses $u_0 \in L^\infty(\mathbf{R}), f \in C^1(\mathbf{R})$. Definition 2 can be extended to more general equations than (1), for example,

$$u_t + f(u, x, t)_x + g(u, x, t) = 0, \tag{14}$$

and multidimensional problems (see [4])

$$u_t + \sum_{j=1}^{p} f_j(u, x, t)x_j + g(u, x, t) = 0 \quad \text{if } (x, t) \in \mathbf{R}^p \times]0, T[. \tag{15}$$

2. This section deals with the numerical approximation of the solution in Kružkov's sense or of a weak solution of (1), (2), by a family of finite difference schemes. As in Conway and Smoller [1], by using the locally bounded variation of u_0 and a result of compactness, we can prove the convergence (of a subsequence) to a weak solution. By imposing an additional condition, we verify that the weak solution obtained at the limit is the solution in Kružkov's sense and using a counterexample, we see that such a condition is necessary.

Let $h > 0$ be the mesh size in space; h is destined to tend to zero and we may suppose it to be bounded ($h \leqslant h_0$). The mesh size in time is $\Delta t = qh$ where q is a constant positive coefficient. $\mathbf{R}$ is divided into an infinity of intervals of length h, $I_i = [(i - ½)h, (i + ½)h]$, for $i \in \mathbf{Z}$.

Let N be the integer part of $T/\Delta t$, the interval $[0, T[$ is covered by $N + 2$ disjointed intervals $J_0 = [0, \Delta t/2[, J_n = [(n - ½)\Delta t, (n + ½)\Delta t[$, with $n \in \{1, \ldots, N + 1\}$. The initial condition $u_0 \in L^\infty(\mathbf{R})$ is supposed to be of locally bounded variation on $\mathbf{R}$, and therefore verifies, for all real δ,

$$\forall R \geqslant 0, \quad \int_{|x|<R} |u_0(x + \delta) - u_0(x)|\, dx \leqslant C(R)|\delta|, \tag{16}$$

where C is an increasing function on $[0, \infty[$, independent of δ. u_0 is approached on each interval I_i by the constant

$$u_i^0 = \frac{1}{h} \int_{I_i} u_0(x)\, dx. \tag{17}$$

The approximate solution u_h is now defined on $\mathbf{R} \times]0, T[$ by $u_h(x, t) = u_i^n$ if $(x, t) \in I_i \times J_n$, with the help of a finite difference scheme of the form

$$u_i^{n+1} = u_i^n - \frac{q}{2}\,[f(u_{i+1}^n) - f(u_{i-1}^n)] + \frac{a_{i+1/2}^n}{2}(u_{i+1}^n - u_i^n) - \frac{a_{i-1/2}^n}{2}(u_i^n - u_{i-1}^n), \tag{18}$$

with $i \in \mathbf{Z}$, $n \leqslant N$, and where the coefficients $a_{i+1/2}^n$ are introduced so that they locally bound the influence of viscosity, and a priori depend on u_i^n and u_{i+1}^n. However, we must specify that the stability and convergence of the scheme depend on the choice of $a_{i+1/2}^n$. The last two terms of (18) locally contribute to "numerical viscosity" of parameter $a_{i+1/2}^n$ on $I_i \times J_n$ which should be compared to the second member of (4).

For $i \in \mathbf{Z}$, $n \in \{0, \ldots, N\}$, we define the closed interval

$$\Gamma^n_{i+1/2} = [\mathrm{Inf}(u^n_i, u^n_{i+1}), \mathrm{Sup}(u^n_i, u^n_{i+1})]\,; \tag{19}$$

and since $f \in C^1(\mathbf{R})$, there exists $\xi^n_{i+1/2} \in \Gamma^n_{i+1/2}$ such that

$$f(u^n_{i+1}) - f(u^n_i) = f'(\xi^n_{i+1/2})(u^n_{i+1} - u^n_i). \tag{20}$$

The following theorem states a first result of convergence.

THEOREM 1. *If the stability condition of Courant-Friedrichs-Lewy*

$$q \cdot \sup_{|\xi| \leqslant |u_0|_{L^\infty(\mathbf{R})}} |f'(\xi)| \leqslant 1 \tag{21}$$

is verified, and if for all $h > 0$, the choice of coefficients $a^n_{i+1/2}$ is such that

$$\forall i \in \mathbf{Z},\ \forall n \leqslant N, \quad q \cdot |f'(\xi^n_{i+1/2})| \leqslant a^n_{i+1/2} \leqslant 1, \tag{22}$$

then the family $\{u_h\}_{h>0}$ contains a convergent sequence in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to a weak solution of (1), (2).

Proof. We establish the same estimates as Conway and Smoller [1].

(a) *Conservation of Stability.* Let $n \in \{0, \ldots, N\}$; if $\mathrm{Sup}_{j\in\mathbf{Z}} |u^n_j| \leqslant |u_0|_{L^\infty(\mathbf{R})}$, we prove that

$$\forall i \in \mathbf{Z}, \quad \mathrm{Inf}(u^n_{i-1}, u^n_i, u^n_{i+1}) \leqslant u^{n+1}_i \leqslant \mathrm{Sup}(u^n_{i-1}, u^n_i, u^n_{i+1}). \tag{23}$$

By introducing (29) into (18), it follows that

$$\begin{aligned} u^{n+1}_i &= u^n_i + (u^n_{i+1} - u^n_i)(a^n_{i+1/2} - qf'(\xi^n_{i+1/2}))/2 \\ &\quad + (u^n_{i-1} - u^n_i)(a^n_{i-1/2} + qf'(\xi^n_{i-1/2}))/2, \end{aligned} \tag{24}$$

where from (21) and (22)

$$\forall j \in \mathbf{Z}, \quad 0 \leqslant (a^n_{j+1/2} \pm qf'(\xi^n_{j+1/2}))/2 \leqslant 1. \tag{25}$$

Let $i \in \mathbf{Z}$. There are six cases corresponding to the different possibilities; we have to classify u^n_{i+1}, u^n_i and u^n_{i-1} in decreasing order.

Case 1. If $u^n_{i+1} \geqslant u^n_i \geqslant u^n_{i-1}$, we have from (24) and (25),

$$u^{n+1}_i \leqslant u^n_i + (u^n_{i+1} - u^n_i) \cdot 1 + (u^n_{i-1} - u^n_i) \cdot 0 = u^n_{i+1},$$

$$u^{n+1}_i \geqslant u^n_i + (u^n_{i+1} - u^n_i) \cdot 0 + (u^n_{i-1} - u^n_i) \cdot 1 = u^n_{i-1},$$

and (23) is proved in this case.

Case 2. If $u^n_{i+1} \geqslant u^n_{i-1} \geqslant u^n_i$, there exists a real ξ^n_i between u^n_{i-1} and u^n_{i+1}, such that

$$f(u^n_{i+1}) - f(u^n_{i-1}) = f'(\xi^n_i)(u^n_{i+1} - u^n_{i-1}), \tag{26}$$

which allows us to give (18) the form

$$u_i^{n+1} = u_{i-1}^n(a_{i-1/2}^n + qf'(\xi_i^n))/2 + u_i^n(1 - (a_{i+1/2}^n + a_{i-1/2}^n)/2)$$
$$\tag{27} + u_{i+1}^n(a_{i+1/2}^n - qf'(\xi_i^n))/2.$$

From (22) the coefficient of u_i^n is nonnegative, and since $u_i^n \leqslant u_{i-1}^n$, we get

$$u_i^{n+1} \leqslant u_{i-1}^n\,[1 - (a_{i+1/2}^n - qf'(\xi_i^n))/2] + u_{i+1}^n(a_{i+1/2}^n - qf'(\xi_i^n))/2.$$

In the same way, from (21) and (22), the coefficient of u_{i-1}^n is nonnegative and $u_{i-1}^n \leqslant u_{i+1}^n$. It follows that $u_i^{n+1} \leqslant u_{i+1}^n$. We also have from (24), (25),

$$u_i^{n+1} \geqslant u_i^n + (u_{i+1}^n - u_i^n)\cdot 0 + (u_{i-1}^n - u_i^n)\cdot 0 = u_i^n;$$

thus (23) is proved.

Each of the four remaining cases may be treated as one of these two cases.

(b) *Conservation of Bounded Variation.* Let $n \leqslant N$ and $I \in \mathbf{N}$; writing (24) for $i \in \{-I, \dots, I\}$ and for $i + 1$, and subtracting, we get

$$u_{i+1}^{n+1} - u_i^{n+1} = (u_{i+2}^n - u_{i+1}^n)(a_{i+3/2}^n - qf'(\xi_{i+3/2}^n))/2 + (u_{i+1}^n - u_i^n)(1 - a_{i+1/2}^n)$$
$$+ (u_i^n - u_{i+1}^n)(a_{i-1/2}^n + qf'(\xi_{i-1/2}^n))/2.$$

All the coefficients are nonnegative; we take absolute values, sum for $i \in \{-I, \dots, I\}$ and group terms. It follows that

$$\sum_{|i|\leqslant I} |u_{i+1}^{n+1} - u_i^{n+1}|$$
$$\leqslant \sum_{|i|\leqslant I} |u_{i+1}^n - u_i^n|\{1 - a_{i+1/2}^n + (a_{i+1/2}^n - qf'(\xi_{i+1/2}^n) + a_{i+1/2}^n + qf'(\xi_{i+1/2}^n))/2\}$$
$$+ |u_{I+2}^n - u_{I+1}^n| + |u_{-I}^n - u_{-I-1}^n|,$$

from which we deduce the conservation of bounded variation in space

$$\tag{28} \sum_{|i|\leqslant I} |u_{i+1}^{n+1} - u_i^{n+1}| \leqslant \sum_{|i|\leqslant I+1} |u_{i+1}^n - u_i^n|.$$

For $i \in \{-I, \dots, I\}$, (24) can be written

$$u_i^{n+1} - u_i^n = (u_{i+1}^n - u_i^n)(a_{i+1/2}^n - qf'(\xi_{i+1/2}^n))/2$$
$$+ (u_{i-1}^n - u_i^n)(a_{i-1/2}^n + qf'(\xi_{i-1/2}^n))/2$$

where the coefficients are nonnegative.

It follows that

$$\sum_{|i|\leqslant I} |u_i^{n+1} - u_i^n| \leqslant \sum_{|i|\leqslant I} |u_{i+1}^n - u_i^n|(a_{i+1/2}^n) - qf'(\xi_{i+1/2}^n) + a_{i+1/2}^n + qf'(\xi_{i+1/2}^n))/2$$
$$+ |u_{-I} - u_{-I-1}|.$$

From (22), $a^n_{i+1/2} \leqslant 1$ for each i; therefore

$$\sum_{|i|\leqslant I} |u_i^{n+1} - u_i^n| \leqslant \sum_{|i|\leqslant I+1} |u_{i+1}^n - u_i^n|. \tag{29}$$

(c) *Convergence.* Let $h > 0$, $n \leqslant N$; if $\operatorname{Sup}_{i\in\mathbf{Z}} |u_i^n| \leqslant |u_0|_{L^\infty(\mathbf{R})}$, we have from (23),

$$\operatorname*{Sup}_{i\in\mathbf{Z}} |u_i^{n+1}| \leqslant \operatorname*{Sup}_{i\in\mathbf{Z}} |u_i^n| \leqslant |u_0|_{L^\infty(\mathbf{R})}$$

Now from (17), $\operatorname{Sup}_{i\in\mathbf{Z}} |u_i^0| \leqslant |u_0|_{L^\infty(\mathbf{R})}$; therefore step by step

$$|u_h|_{L^\infty(\mathbf{R}\times]0,T[)} \leqslant \operatorname*{Sup}_{i,n} |u_i^n| \leqslant |u_0|_{L^\infty(\mathbf{R})}; \tag{30}$$

and thus, the family $\{u_h\}$ contains a subsequence $\{u_{h^*_m}\}$ weakly-star convergent to a function $u \in L^\infty(\mathbf{R} \times]0, T[)$, bounded by $|u_0|_{L^\infty(\mathbf{R})}$.

Let v_h be the interpolate of degree one of u_h at the vertices of each rectangle $[ih, (i+1)h] \times [nqh, (n+1)qh]$; v_h is continuous, uniformly bounded by $|u_0|_{L^\infty(\mathbf{R})}$ in h, and differentiable inside each rectangle, where it is given by

$$\begin{aligned} v_h(x, t) = u_i^n &+ (u_{i+1}^n - u_i^n)\frac{x - ih}{h} + (u_i^{n+1} - u_i^n)\frac{t - nqh}{qh} \\ &+ (u_{i+1}^{n+1} - u_i^{n+1} - u_{i+1}^n + u_i^n)\frac{x - ih}{h}\,\frac{t - nqh}{ah}. \end{aligned} \tag{31}$$

Let $R > 0$; we denote by I_h the smallest integer such that

$$\Omega_R =]-R, R[\times]0, T[\subset \{(x, t) | (x, t) \in I_i \times J_n \Rightarrow |i| + n < I_h\}.$$

This choice implies that $hI_h \longrightarrow R + T/q$, and remains uniformly bounded ($hI_h \leqslant R_0$).

Thus, we obtain, successively

$$\begin{aligned} \iint_{\Omega_R} |v_{h,t}|\, dx\, dt &\leqslant \sum_{n=0}^{N} \sum_{|i|\leqslant I_h-n-1} |u_i^{n+1} - u_i^n| h \quad \text{from (31)}, \\ &\leqslant \sum_{n=0}^{N} \sum_{|i|\leqslant I_h} |u_{i+1}^0 - u_i^0| h \quad \text{from (29) and (28) } n \text{ times}, \\ &\leqslant (N+1) \sum_{|i|\leqslant I_h} \int_{I_i} |u_0(x+h) - u_0(x)|\, dx \quad \text{from (17)}, \\ &\leqslant (h_0 + T/q) C(R_0) \quad \text{from (16) with } \delta = h. \end{aligned}$$

In the same way:

$$\iint_{\Omega_R} |v_{h,x}|\, dx\, dt \leqslant \sum_{n=0}^{N+1} \sum_{|i|\leqslant I_h-n} |u_{i+1}^n - u_i^n| qh \leqslant (2qh_0 + T) C(R_0).$$

By putting $M = |u_0|_{L^\infty(\mathbf{R})} + (2h_0 + T/q)C(R_0)(1 + q)$ (independent of h), it follows that

$$|v_h|_{L^\infty(\Omega_R)} + |v_{h,x}|_{L^1(\Omega_R)} + |v_{h,t}|_{L^1(\Omega_R)} \leqslant M. \tag{32}$$

Therefore, from $\{v_{h_m^*}\}$, associated to $\{u_{h_m^*}\}$, we can extract a subsequence $\{v_{h_{m,R}}\}$ convergent in $L^1(\Omega_R)$, (see [1]), from which, with the same argument, we can once more extract a subsequence $\{v_{h_{m,R+1}}\}$ convergent in $L^1(\Omega_{R+1})$. Step by step, from $\{v_{h_{m,R+j-1}}\}$ we extract $\{v_{h_{m,R+j}}\}$ convergent in $L^1(\Omega_{R+j})$. By the diagonal process, which consists in keeping only index $h_m = h_{m,R+m}$, at last we extract a subsequence $\{v_h\}$ convergent in $L^1_{\text{loc}}(\mathbf{R} \times]0, T[)$. Since $\{u_{h_m}\} \subset \{u_{h_m^*}\}$ which weakly star converges to $u \in L^\infty(\mathbf{R} \times]0, T[)$ and by using (28) and (29), as previously, we verify that $\{u_{h_m} - v_{h_m}\}$ tends to zero in $L^1(\Omega)$, for all bounded open sets $\Omega \subset \mathbf{R} \times]0, T[$.

Since v_{h_m} converges in $L^1_{\text{loc}}(\mathbf{R} \times]0, T[)$, finally we have

$$u_{h_m} \xrightarrow{L^1_{\text{loc}}(\mathbf{R}\times]0,T[)} u \in L^\infty(\mathbf{R} \times]0, T[). \tag{33}$$

(d) The limit is a weak solution of (1), (2). Since $u \in L^\infty(\mathbf{R} \times]0, T[)$, now we have to verify (3). Let $\phi \in C_0^2(\mathbf{R} \times [0, T[)$; we multiply (18) by $\phi_i^n = \phi(ih, nqh)$ and by h, then we sum for $n \leqslant N$ and $i \in \mathbf{Z}$ (actually $|i| \leqslant I$, since ϕ has a compact support).

Since

$$\sum_{n=0}^{N} (u_i^{n+1} - u_i^n)\phi_i^n = -\sum_{n=1}^{N+1} u_i^n \frac{\phi_i^n - \phi_i^{n-1}}{qh} qh - u_i^0 \phi_i^0,$$

by introducing $\phi_i^{N+1} = 0$ and treating the other terms in the same way, we get

$$\begin{aligned}\sum_{n=1}^{N+1} \sum_{i\in\mathbf{Z}} u_i^n \frac{\phi_i^n - \phi_i^{n-1}}{qh} qh^2 + \sum_{n=0}^{N} \sum_{i\in\mathbf{Z}} f(u_i^n) \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h} qh^2 + \sum_{i\in\mathbf{Z}} u_i^0 \phi_i^0 h \\ = \frac{1}{2q} \sum_{n=0}^{N} \sum_{i\in\mathbf{Z}} d_{i+1/2}^n (u_{i+1}^n - u_i^n) \frac{\phi_{i+1}^n - \phi_i^n}{h} qh^2.\end{aligned} \tag{34}$$

The first member of (34) tends to the first member of (3), when $h \in \{h_m\}$ tends to zero, since the strong convergence allows us to treat the second nonlinear term (see Oleĭnik [9]). The second member tends to zero; let $R \in \mathbf{R}$ such that Support$(\phi) \subset [-R, R]$, then $hI \leqslant R$, and

$$\begin{aligned}\left|\sum_{n=0}^{N} \sum_{i\in\mathbf{Z}} d_{i+1/2}^n (u_{i+1}^n - u_i^n) \frac{\phi_{i+1}^n - \phi_i^n}{h} qh^2\right| &\leqslant \sum_{n=0}^{N} \sum_{|i|\leqslant I} |u_{i+1}^n - u_i^n| \left|\frac{\partial\phi}{\partial x}\right|_{L^\infty(\mathbf{R}\times]0,T[)} qh^2 \\ &\leqslant \left|\frac{\partial\phi}{\partial x}\right|_{L^\infty} \cdot (N+1) \sum_{|i|\leqslant I+N} |u_{i+1}^0 - u_i^0| qh^2 \quad \text{from (28)}, \\ &\leqslant \left[\left|\frac{\partial\phi}{\partial x}\right|_{L^\infty} (T + qh_0) C(R + T/q)\right] h \quad \text{from (17) and (16).}\end{aligned}$$

Thus $\{u_{h_m}\}$ converges in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to a weak solution of (1), (2), and Theorem 1 is proved.

In order to assure that the limit u is the solution in Kružkov's sense of (1), (2), we must furthermore restrict the choice of coefficients $\{a^n_{i+1/2}\}$, as the following counterexample shows. By numerically solving Example 3 with a scheme of the form (18), where the choice of coefficients verifies

$$(35)\qquad \forall n \leqslant N, \forall i \in Z, \quad a^n_{i+1/2} = q|f'(\xi^n_{i+1/2})|,$$

the family $\{u_h\}$ converges to the weak solution u_1, which does not satisfy the entropy condition (7). Indeed, from (17), the discrete initial condition is $u^0_i = sg(i)$, and thus $f(u^0_i)$ is zero for all $i \in \mathbf{Z}$. We have $a^0_{i+1/2}(u^0_{i+1} - u^0_i) = 0$, and thus the scheme is reduced to $u^1_i = u^0_i$, and step by step $u^{n+1}_i = u^n_i = \cdots = u^0_i$. Obviously,

$$|u_h - u_1|_{L^1(\Omega_R)} \leqslant Th \quad \text{for all } R > 0,$$

and $\{u_h\}$ converges in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to u_1 which differs from the solution in Kružkov's sense, u_2.

Let us note that if (35) is only verified for $i = -1$ and $i = 0$, then we get the same conclusions. By solving Example 1 with the numerical initial condition $u^0_i = -1$ if $i \leqslant 0$, and 1 if $i > 1$, which is the same as replacing (17) by

$$u^0_i = (1/h)\int_{ih}^{(i+1)h} u_0(x)\,dx,$$

we have convergence to u_1, which does not satisfy (7).

By introducing, for all $i \in \mathbf{Z}$ and $n \leqslant N$, the quantity

$$(36)\quad s^n_{i+1/2} = \begin{cases} |f'(\xi^n_{i+1/2})| & \text{if } u^n_{i+1} = u^n_i, \\[2ex] \displaystyle\operatorname*{Sup}_{k \in \Gamma^n_{i+1/2}} \left[\operatorname{Max}\left\{\frac{f(u^n_{i+1}) - f(k)}{u^n_{i+1} - k}, -\frac{f(u^n_i) - f(k)}{u^n_i - k}\right\}\right] & \text{if } u^n_{i+1} \neq u^n_i, \end{cases}$$

we get a result of convergence to the solution in Kružkov's sense of (1), (2).

THEOREM 2. *If the stability condition of Courant-Friedrichs-Lewy* (21) *is verified, and if for all $h > 0$, the choice of coefficients $\{a^n_{i+1/2}\}$ is such that*

$$(37)\qquad \forall i \in \mathbf{Z}, \forall n \leqslant N, \quad q s^n_{i+1/2} \leqslant a^n_{i+1/2} \leqslant 1,$$

then the family $\{u_h\}$ converges in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to the solution in Kružkov's sense of (1), (2).

Proof. Let $i \in \mathbf{Z}$, $n \leqslant N$; by successively taking $k = u^n_i$ and $k = u^n_{i+1}$ in (36), we verify that $q|f'(\xi^n_{i+1/2})| \leqslant q s^n_{i+1/2} \leqslant 1$, so that the choice of $a^n_{i+1/2}$ is always possible and Theorem 1 can be used. Therefore, $\{u_h\}$ contains a sequence $\{u_{h_m}\}$ convergent in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to a weak solution u of (1), (2). If u satisfies Definition 2, then the uniqueness of the solution in Kružkov's sense implies the convergence of the whole family $\{u_h\}$.

(a) *Verification of* (i). Let $k \in \mathbf{R}$, $h > 0$, $i \in \mathbf{Z}$, $n \leqslant N$; we first prove

$$
\begin{aligned}
|u_i^{n+1} - k| \leqslant |u_i^n - k| & \\
- \frac{q}{2}\{ [f(u_{i+1}^n) - f(k)]\, sg(u_{i+1}^n - k) - [f(u_{i-1}^n) - f(k)]\, sg(u_{i-1}^n - k)\} & \qquad (38)\\
+ \frac{a_{i+1/2}^n}{2}(|u_{i+1}^n - k| - |u_i^n - k|) - \frac{a_{i-1/2}^n}{2}(|u_i^n - k| - |u_{i-1}^n - k|), &
\end{aligned}
$$

case by case and using k_i^n defined by

(39) $\forall i \in \mathbf{Z}, \quad \exists k_i^n \in [\mathrm{Inf}(u_i^n, k), \mathrm{Sup}(u_i^n, k)], \qquad f(u_i^n) - f(k) = f'(k_i^n)(u_i^n - k).$

Case 0. $k > \mathrm{Sup}\{u_{i-1}^n, u_i^n, u_{i+1}^n\}$ or $k < \mathrm{Inf}\{u_{i-1}^n, u_i^n, u_{i+1}^n\}$. We have from (18)

$$
\begin{aligned}
u_i^{n+1} - k = u_i^n - k - \frac{q}{2}\,[(f(u_{i+1}^n) - f(k)) - (f(u_{i-1}^n) - f(k))] & \\
+ \frac{a_{i+1/2}^n}{2}\,[(u_{i+1}^n - k) - (u_i^n - k)] - \frac{a_{i-1/2}^n}{2}\,[(u_i^n - k) - (u_{i-1}^n - k)], &
\end{aligned}
$$

and we multiply by $s = sg(u_i^{n+1} - k)$ $(= sg(u_{i-1}^n - k) = sg(u_i^n - k) = sg(u_{i+1}^n - k))$, from (23).

Case 1. $u_{i+1}^n > k > u_i^n > u_{i-1}^n$ or $u_{i+1}^n < k < u_i^n < u_{i-1}^n$. We subtract k from each member of (18), and we put $u_{i+1}^n - u_i^n = (u_{i+1}^n - k) - (u_i^n - k)$. From (39),

$$f(u_{i+1}^n) - f(u_{i+1}^n) = f'(k_{i+1}^n)(u_{i+1}^n - k) - f'(k_i^n)(u_i^n - k) + f'(\xi_{i-1/2}^n)(u_i^n - u_{i-1}^n).$$

Thus we have

$$
\begin{aligned}
u_i^{n+1} - k = (u_{i+1}^n - k)(a_{i+1/2}^n - qf'(k_{i+1}^n))/2 + (u_i^n - k)(1 + (qf'(k_i^n) - a_{i+1/2}^n)/2) & \\
+ (u_{i-1}^n - u_i^n)(a_{i-1/2}^n + qf'(\xi_{i-1/2}^n))/2. &
\end{aligned}
$$

From (22), (36) and (37), since $k \in \Gamma_{i+1/2}^n$, coefficients are nonnegative; therefore

$$
\begin{aligned}
|u_i^{n+1} - k| \leqslant |u_{i+1}^n - k|(a_{i+1/2}^n - qf'(k_{i+1}^n))/2 + |u_i^n - k|(1 + (qf'(k_i^n) - a_{i+1/2}^n)/2) & \\
+ |u_i^n - u_{i-1}^n|(a_{i-1/2}^n + qf'(\xi_{i-1/2}^n))/2. &
\end{aligned}
$$

We now group terms; since $\forall j \in \mathbf{Z}$, $|u_j^n - k| f'(k_j^n) = (f(u_j^n) - f(k))sg(u_j^n - k)$, it follows that

$$
\begin{aligned}
|u_i^{n+1} - k| \leqslant |u_i^n - k| - \frac{q}{2}\,[(f(u_{i+1}^n) - f(k))sg(u_{i+1}^n - k) - (f(u_i^n) - f(k))sg(u_i^n - k)] & \\
+ a_{i+1/2}^n(|u_{i+1}^n - k| - |u_i^n - k|)/2 + |u_i^n - u_{i-1}^n|(a_{i-1/2}^n + qf'(\xi_{i-1/2}^n))/2. &
\end{aligned}
$$

But $sg(u_i^n - k) = sg(u_{i-1}^n - k) = sg(u_{i-1}^n - u_i^n)$, therefore

$$a_{i-1/2}^n |u_i^n - u_{i-1}^n| = a_{i-1/2}^n (u_{i-1}^n - k + k - u_i^n) sg(u_{i-1}^n - u_i^n)$$
$$= - a_{i-1/2}^n (|u_i^n - k| - |u_{i-1}^n - k|),$$

and

$$|u_i^n - u_{i-1}^n| f'(\xi_{i-1/2}^n) = \{ [f(u_i^n) - f(k)] sg(u_i^n - k) - f(u_{i-1}^n) - f(k) sg(u_{i-1}^n - k)\}.$$

Thus, we obtain (38).

Case 2. $u_{i+1}^n > u_i^n > k > u_{i-1}^n$ or $u_{i+1}^n < u_i^n < k < u_{i-1}^n$. We have the same kind of argument as previously. We write this time

$$f(u_{i+1}^n) - f(u_{i-1}^n) = f'(\xi_{i+1/2}^n)(u_{i+1}^n - u_i^n) + f'(k_i^n)(u_i^n - k) - f'(k_{i-1}^n)(u_{i-1}^n - k),$$

and $u_i^n - u_{i-1}^n = (u_i^n - k) - (u_{i-1}^n - k)$; then (18) is changed into

$$u_i^{n+1} - k = (u_{i+1}^n - u_i^n)(a_{i+1/2}^n - qf'(\xi_{i+1/2}^n))/2 + (u_i^n - k)(1 - (a_{i-1/2}^n + qf'(k_i^n))/2)$$
$$+ (u_{i-1}^n - k)(a_{i-1/2}^n + qf'(k_{i-1}^n))/2,$$

where all the coefficients are nonnegative from (21), (22), (36) and (37), because $k \in \Gamma_{i+1/2}^n$. Taking absolute values and observing that $sg(u_{i+1}^n - u_i^n) = sg(u_i^n - k) = sg(u_{i+1}^n - k)$, we get

$$|u_{i+1}^n - u_i^n|(a_{i+1/2}^n - qf'(\xi_{i+1/2}^n))$$
$$= a_{i+1/2}^n(|u_{i+1}^n - k| - |u_i^n - k|)$$
$$- q(f(u_{i+1}^n) - f(k)) sg(u_{i+1}^n - k) + q(f(u_i^n) - f(k)) sg(u_i^n - k).$$

From that we deduce (38).

Case 3. $u_{i+1}^n > k > u_{i-1}^n > u_i^n$ or $u_{i+1}^n < k < u_{i-1}^n < u_i^n$. Since

$$f(u_{i+1}^n) - f(u_{i-1}^n) = f(u_{i+1}^n) - f(k) - (f(u_{i-1}^n) - f(k))$$
$$= f'(k_{i+1}^n)(u_{i+1}^n - k) - f'(k_{i-1}^n)(u_{i-1}^n - k),$$

we obtain from (18)

$$u_i^{n+1} - k = u_i^n - k - \frac{q}{2}[f'(k_{i+1}^n)(u_{i+1}^n - k) - f'(k_{i-1}^n)(u_{i-1}^n - k)]$$
$$+ a_{i+1/2}((u_{i+1}^n - k) - (u_i^n - k))/2 - a_{i-1/2}^n((u_i^n - k) - (u_{i-1}^n - k))/2.$$

By grouping terms, it follows that

$$u_i^{n+1} - k = (u_{i-1}^n - k)(a_{i-1/2}^n + qf'(k_{i-1}^n))/2$$
$$+ (u_i^n - k)(1 - (a_{i+1/2}^n + a_{i-1/2}^n)/2) + (u_{i+1}^n - k)(a_{i+1/2}^n - qf'(k_{i+1}^n))/2. \tag{40}$$

Since $u_i^n - k = u_i^n - u_{i-1}^n + u_{i-1}^n - k$, we obtain

$$u_i^{n+1} - k = (u_{i-1}^n - k)\left(1 - \frac{a_{i-1/2}^n}{2} + \frac{q}{2}f'(k_{i-1}^n)\right)$$

$$+ (u_i^n - u_{i-1}^n)(1 - (a_{i+1/2}^n + a_{i-1/2}^n)/2) + (u_{i+1}^n - k)(a_{i+1/2}^n - qf'(k_{i+1}^n))/2,$$

where, from (21), (36), (37) ($k \in \Gamma_{i+1/2}^n$), coefficients are nonnegative. Taking absolute values and noting that $|u_i^n - u_{i-1}^n| = |u_i^n - k| - |u_{i-1}^n - k|$, we obtain

$$\begin{aligned} |u_i^{n+1} - k| \leqslant{} & |u_{i-1}^n - k|(a_{i-1/2}^n + qf'(k_{i-1}^n))/2 \\ & + |u_i^n - k|(1 - (a_{i+1/2}^n + a_{i-1/2}^n)/2) \\ & + |u_{i+1}^n - k|(a_{i+1/2}^n - qf'(k_{i+1}^n))/2. \end{aligned} \tag{41}$$

If we group terms inversely to (40), we recognize (38) exactly.

Case 4. $u_{i+1}^n > u_{i-1}^n > k > u_i^n$ or $u_{i+1}^n < u_{i-1}^n < k < u_i^n$. As before, we change (18) into (40), where all the coefficients are nonnegative, since $k \in \Gamma_{i+1/2}^n \cap \Gamma_{i-1/2}^n$. Therefore, we get (41) directly, i.e. (38).

Case 5. $u_i^n > k > u_{i+1}^n > u_{i-1}^n$ or $u_i^n < k < u_{i+1}^n < u_{i-1}^n$. As in Case 4, we change (18) into (40) where coefficients are nonnegative, since $k \in \Gamma_{i+1/2}^n \cap \Gamma_{i-1/2}^n$; from that, we deduce (41) directly, i.e. (38).

Case 6. $u_i^n > u_{i+1}^n > k > u_{i-1}^n$ or $u_i^n < u_{i+1}^n < k < u_{i-1}^n$. (18) is changed into (40), where we put $u_i^n - k = u_i^n - u_{i+1}^n + u_{i+1}^n - k$. It follows that

$$u_i^{n+1} - k = (u_{i-1}^n - k)(a_{i-1/2}^n + qf'(k_{i-1}^n))/2 + (u_i^n - u_{i+1}^n)(1 - (a_{i+1/2}^n + a_{i-1/2}^n)/2)$$

$$+ (u_{i+1}^n - k)\left(1 - \frac{a_{i+1/2}^n}{2} - \frac{q}{2}f'(k_{i+1}^n)\right),$$

where coefficients are nonnegative. We take absolute values, and noting that $|u_i^n - u_{i+1}^n| = |u_i^n - k| - |u_{i+1}^n - k|$, we find again (41), hence (38).

Case 7. $u_{i+1}^n > k > u_i^n = u_{i-1}^n$ or $u_{i+1}^n < k < u_i^n = u_{i-1}^n$. We have from (18)

$$u_i^{n+1} - k = (u_i^n - k)(1 - (a_{i+1/2}^n - qf'(k_{i-1}^n))/2)$$

$$+ (u_{i+1}^n - k)(a_{i+1/2}^n - qf'(k_{i+1}^n))/2,$$

where coefficients are nonnegative. We take absolute values, we group by using $u_i^n = u_{i-1}^n$ and (39) and we get (38).

Case 8. $u_{i+1}^n = u_i^n > k > u_{i-1}^n$ or $u_{i+1}^n = u_i^n < k < u_{i-1}^n$. The same argument as before applies, but we exchange parts of u_{i-1}^n and u_{i+1}^n.

Case 9. $u_{i+1}^n = u_{i-1}^n > k > u_i^n$ or $u_{i+1}^n = u_{i-1}^n < k < u_i^n$. The same argument as in Cases 4 and 5 applies, where $sg(u_{i+1}^n - u_{i-1}^n)$ has not been used.

After these ten cases, (38) is verified for all $k \notin \{u_{i-1}^n, u_i^n, u_{i+1}^n\}$. Since (38) is made up of continuous functions in the variable k, it is verified for all real k, and thus (38) is proved.

Let $\phi \in C_0^2(\mathbf{R} \times]0, T[)$ be nonnegative; we multiply (38) by h and $\phi_i^n = \phi(ih, nqh)$, which keeps inequality. Then, we sum for $i \in \mathbf{Z}$ (actually $|i| \leqslant I$, if we consider the support of ϕ) and with the same process as in the proof of Theorem 1 (part (d)) we obtain a formulation similar to (34)

$$\sum_{n=1}^{N+1} \sum_{i \in \mathbf{Z}}^{n=1} \left\{ |u_i^n - k| \frac{\phi_i^n - \phi_i^{n-1}}{qh} + sg(u_i^n - k)(f(u_i^n) - f(k)) \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h} \right\} qh^2 \tag{42}$$
$$\geqslant \sum_{n=1}^{N} \sum_{|i| \leqslant I} \frac{a_{i+1/2}^n}{2q} (|u_{i+1}^n - k| - |u_i^n - k|) \frac{\phi_{i+1} - \phi_i^n}{h} qh^2.$$

Now $|u - k|$ and $sg(u - k)(f(u) - f(k))$ are Lipschitz continuous functions of u; if we only consider the terms of the sequence $\{u_{h_m}\}$ convergent in $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ to u, the first member of (42) converges to the first member of (12). The second member tends to zero so we can estimate it by the second member of (34), since $||u_{i+1}^n - k| - |u_i^n - k|| \leqslant |u_{i+1}^n - u_i^n|$. Thus, we obtain (12) and u verifies (i).

(b) *Verification of* (ii). Let $R > 0$; the sequence $\{u_{h_m}\}$ converges in $L^1(\Omega_R)$ to u; and therefore,we can extract a subsequence $\{u_{h'_m}\}$ convergent to u, almost everywhere on Ω_R. According to the Fubini theorem, for almost all $t \in]0, T[$, $u_{h'_m}(\cdot, t)$ converges to $u(\cdot, t)$ almost everywhere on $]-R, R[$. Now $\{u_{h'_m}\}$ remains uniformly bounded; according to the Lebesgue theorem we have

$$\text{for almost all } t \in]0, t[, \quad \lim_{m \to \infty} \int_{|x|<R} |u_{h'_m}(x, t) - u(x, t)|\, dx = 0. \tag{43}$$

Let E be the negligible set of $t \in]0, T[$ for which (43) is not verified; if $t \in]0, T[\backslash E$, for all m we have

$$\begin{aligned} \int_{|x|<R} |u(x, t) - u_0(x)|\, dx \leqslant & \int_{|x|<R} |u(x, t) - u_{h'_m}(x, t)|\, dx \\ & + \int_{|x|<R} |u_{h'_m}(x, t) - u_{h'_m}(x, 0)|\, dx \\ & + \int_{|x|<R} |u_{h'_m}(x, 0) - u_0(x)|\, dx. \end{aligned} \tag{44}$$

We separately study each term of the second member of (44), when m tends to infinity. Since $t \notin E$, from (43), the first term tends to zero. As for the second term, we introduce two integers I and n, such that $(I - 1)h'_m \leqslant R < Ih'_m$, $t \in J_n$. When m tends to infinity, Ih'_m tends to R and nh'_m tends to T/q, and then $(I + n)h'_m$ remains uniformly bounded in h $((I + n)h'_m \leqslant R_0)$. From (29), (28), (17) and (16) we have successively

$$\int_{|x|<R} |u_{h'_m}(x, t) - u_{h'_m}(x, 0)|\, dx$$
$$\leqslant \sum_{|i| \leqslant I} |u_i^n - u_i^0| h'_m \leqslant \sum_{|i| \leqslant I} \sum_{\nu=1}^{n} |u_i^\nu - u_i^{\nu-1}| h'_m \leqslant C(R_0)(nh'_m).$$

We get for the upper limit

$$\overline{\lim_{m\to\infty}} \int_{|x|<R} |u_{h'_m}(x, t) - u_{h'_m}(x, 0)|\, dx \leqslant C(R_0)t/q. \tag{45}$$

For the last term, we introduce the $w_{h'_m}$ interpolate of degree one of $u_{h'_m}(\cdot, 0)$ at the points ih'_m on $]-R, R[$. From (16) and (17), it follows that

$$|w_{h_m}|_{L^\infty(-R,R)} \leqslant |u_0|_{L^\infty(\mathbf{R})}; \qquad |w_{h'_m,x}|_{L^1(-R,R)} \leqslant \sum_{|i|\leqslant I} |u^0_{i+1} - u^0_i| \leqslant C(R_0).$$

Therefore, $\{w_{h'_m}\}$ contains a subsequence $\{w_{h''_m}\}$ convergent in $L^1(-R, R)$ to $w \in L^\infty(-R, R)$, by using the same compactness argument as for Theorem 1 (see [1]). Now we just have to verify that $w = u_0$ almost everywhere, since $\{u_{h''_m} - v_{h''_m}\}$ tends to zero in $L^1(-R, R)$ when m tends to infinity.

Let $\psi \in C^1_0(-R, R)$; we approach ψ by

$$\psi_m(x) = \psi^i = \frac{1}{h''_m} \int_{I_i} \psi(x)\, dx \quad \text{if } x \in I_i.$$

The sequence $\{\psi_m\}$ converges to ψ in $L^1(-R, R)$ when m tends to infinity, and if m is great enough $\psi^I = \psi^{-I} = 0$ (compact support of ψ), thus we have

$$\int_{|x|<R} u_{h''_m}(x, 0)\psi(x)\, dx = \sum_{|i|\leqslant I} u^0_i \psi^i h''_m = \int_{|x|<R} u_0(x)\psi_m(x)\, dx,$$

and $\{u_{h''_m}(\cdot, 0)\}$ weakly converges to u_0. Since a strong convergence to w exists already, $w = u_0$ almost everywhere in $]-R, R[$. Therefore, (44) gives at the limit

$$\int_{|x|<R} |u(x, t) - u_0(x)|\, dx \leqslant C(R_0)t/q \quad \text{if } t \in]0, T[\backslash E,$$

by using (45). From that we can deduce (13) immediately, and thus Theorem 2 is proved.

Theorems 1 and 2 can be generalized for similar numerical schemes in order to solve more general equations such as (14) or (15). Some of these generalizations are studied in [7].

3. In this section we study various applications. We shall give Lax, Godounov and Lax-Wendroff schemes of the form (18), and study them according to Theorems 1 and 2. Then, we shall try to give an interpretation of conditions (22) and (37). At last, we shall consider the particular case when f is monotone (decentered scheme). We denote by (S) the scheme of the form (18) with $a^n_{i+1/2} = qs^n_{i+1/2}$, and by (T) the scheme with $a^n_{i+1/2} = q|f'(\xi^n_{i+1/2})|$.

Lax Scheme. In [5], Lax proposes a scheme written with previous notations as

$$u^{n+1}_i = (u^n_{i+1} + u^n_{i-1})/2 - q[f(u^n_{i+1}) - f(u^n_{i-1})]/2. \tag{46}$$

The Lax scheme can be written with the form (18)

$$u^{n+1}_i = u^n_i - q[f(u^n_{i+1}) - f(u^n_{i-1})]/2 + (u^n_{i+1} - u^n_i)/2 - (u^n_i - u^n_{i-1})/2,$$

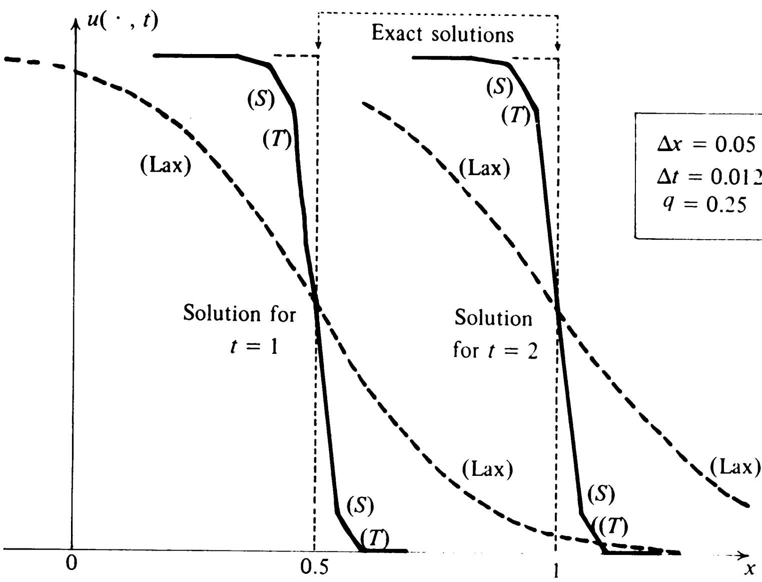
and corresponds to a uniform choice of coefficients

(47) $$\forall i \in \mathbf{Z}, \forall n \leqslant N, \quad a^n_{i+1/2} = 1.$$

THEOREM 3. *If the stability condition of Courant-Friedrichs-Lewy* (21) *is verified, then the family* $\{u_h\}_{h>0}$ *of approximated solutions, built by the Lax scheme, converges in* $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ *to the solution in Kružkov's sense of* (1), (2).

This is not a new result; Conway and Smoller [1] state convergence to a weak solution, and Douglis [2] as well as Oharu-Takahashi [8] prove convergence to the solution in Kružkov's sense. From (47), the Lax scheme contains a uniform numerical viscosity, which has the effect of spreading shocks. The numerical solution of the following problem

$$u_t + uu_x = 0, \qquad u(x, 0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$



(which in Kružkov's sense is $u(x, t) = 1$ if $x < t/2$ and 0 if $x > t/2$), has been computed with the same conditions by the Lax scheme and by schemes (S) and (T) which coincide about this example. The numerical results we obtained are reproduced in the figure above. The Lax scheme gives better results when q is near 1, i.e. when the inequality (21) is satisfied nearer to equality, while the other schemes do not vary slightly. Indeed, if we change (18) into

(48) $$\frac{u_i^{n+1} - u_i^n}{qh} + \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2h} = \frac{1}{h}\left\{\left(\frac{h}{q} a^n_{i+1/2}\right)\frac{u^n_{i+1} - u^n_i}{h} - \left(\frac{h}{q} a^n_{i-1/2}\right)\frac{u^n_i - u^n_{i-1}}{h}\right\}.$$

we notice that the coefficient of numerical viscosity of the Lax scheme is h/q, and therefore increases when q decreases. As for schemes (S) and (T), this coefficient is $hs^n_{i+1/2}$, which is independent of q. If we consider (48), we see that (18) is obtained by multiplying the coefficient of numerical viscosity of the Lax scheme by $a^n_{i+1/2}$.

Godounov Scheme. Here is the way Oleĭnik describes the Godounov scheme in [9]. This scheme is given by

$$u_i^{n+1} = u_i^n - q\,[f(u^n_{i+1/2}) - f(u^n_{i-1/2})], \tag{49}$$

where the quantities $u^n_{j+1/2}$ $(j \in \mathbf{Z})$ are obtained by

$$\begin{cases} \text{if } f'(u_j^n) > 0 \text{ and } f'(\xi^n_{j+1/2}) > 0 & \text{then } u^n_{j+1/2} = u^n_j, \\ \text{if } f'(u^n_{j+1}) < 0 \text{ and } f'(\xi^n_{j+1/2}) < 0 & \text{then } u^n_{j+1/2} = u^n_{j+1}, \\ \text{in other cases, } u^n_{j+1/2} \text{ is the solution in } \Gamma^n_{j+1/2} \text{ of } f'(u) = 0. \end{cases} \tag{50}$$

By applying this scheme to Burgers equation for the (numerical) initial condition $u_i^0 = 1$ if $i \leqslant 0$ and -1 if $i > 0$, we verify that $\mathrm{Sup}_{i\in\mathbf{Z}} |u_i^n| = 1 + n/2$ (see [7]) and observe that an instability appears near the shock. This phenomenon vanishes when we add the following criteria to the choice of $u^n_{j+1/2}$

$$\begin{cases} \text{when } f(u^n_{j+1}) = f(u^n_j), \text{ we first seek } u^n_{j+1/2} \in \Gamma^n_{j+1/2} \text{ as a root of } f', \\ \text{and if: } (u^n_{j+1} - u^n_j)(f(u^n_{j+1/2}) - f(u^n_j)) > 0, \text{ we replace it by } u^n_{j+1/2} = u^n_j. \end{cases} \tag{51}$$

Let us note that by taking $u^n_{j+1/2} = u_{j+1}$, at last we find the same result. (49) can be changed into

$$u_i^{n+1} = u_i^n - \frac{q}{2}\,[f(u^n_{i+1}) - f(u^n_{i-1})] + \frac{q}{2}[f(u^n_{i+1}) - 2f(u^n_{i+1/2}) + f(u^n_i)]$$
$$- \frac{q}{2}\,[f(u^n_i) - 2f(u^n_{i-1/2}) + f(u^n_{i-1})];$$

and to give it the form (18), we must select $a^n_{i+1/2}$ such that

$$a^n_{i+1/2}(u^n_{i+1} - u^n_i) = q\,[f(u^n_{i+1}) - 2f(u^n_{i+1/2}) + f(u^n_i)]. \tag{52}$$

Since $u^n_{i+1/2} \in \Gamma^n_{i+1/2}$, we can find $\lambda \in [0, 1]$ such that

$$u^n_{i+1/2} = \lambda u^n_{i+1} + (1 - \lambda)u^n_i,$$

and two elements η_1 and η_2 of $\Gamma^n_{i+1/2}$ verifying

$$\begin{aligned} f(u^n_{i+1}) - f(u^n_{i+1/2}) &= f'(\eta_1)(u^n_{i+1} - u^n_{i+1/2}); \\ f(u^n_{i+1/2}) - f(u^n_i) &= f'(\eta_2)(u^n_{i+1/2} - u^n_i). \end{aligned} \tag{53}$$

By introducing these quantities in (52), we get when $u_i^n \neq u_{i+1}^n$,

$$a_{i+1/2}^n = q[(1-\lambda)f'(\eta_1) - \lambda f'(\eta_2)]. \tag{54}$$

If $u_{i+1}^n = u_i^n$, then we may take $a_{i+1/2}^n = q|f'(\xi_{i+1/2}^n)|$ $(= q|f'(u_i^n)|)$. By comparing (36) with (53), it follows that $f'(\eta_1) \leqslant s_{i+1/2}^n$ and $-f'(\eta_2) \leqslant s_{i+1/2}^n$; therefore $a_{i+1/2}^n \leqslant q|1 - \lambda + \lambda|s_{i+1/2}^n = qs_{i+1/2}^n$.

Then, it may happen that Theorem 2 cannot be applied. Nevertheless, Theorem 1 is applicable, provided that we choose the root of f' properly, when $f'(\xi_{i+1/2})$ is not zero.

THEOREM 4. *If the stability condition of Courant-Friedrichs-Lewy* (21) *is verified, then the family* $\{u_h\}$ *of approximate solutions, built by the* (*revised*) *Godounov scheme, contains a sequence* $\{u_{h_m}\}_m$ *convergent in* $L^1_{\mathrm{loc}}(\mathbf{R} \times]0, T[)$ *to a weak solution of problem* (1), (2).

Proof. It is sufficient to verify (22). If $u_{i+1}^n = u_i^n$, the result is obvious. If $u_{i+1}^n \neq u_i^n$, from (54) and (21), $a_{i+1/2}^n \leqslant 1$.

.If $f'(\xi_{i+1/2}^n) > 0$ and $f'(u_i^n) > 0$, then $u_{i+1/2}^n = u_i^n$, and from (52), $a_{i+1/2}^n = qf'(\xi_{i+1/2}^n) > 0$.

.If $f'(\xi_{i+1/2}^n) > 0$ and $f'(u_i^n) \leqslant 0$, we choose for $u_{i+1/2}^n$, the root of f' the nearest of u_i^n, so that $f'(\eta_2) \leqslant 0$. Then, from (52)

$$a_{i+1/2}^n(u_{i+1}^n - u_i^n) = q[f(u_{i+1}^n) - f(u_i^n)] - 2q[f(u_{i+1/2}^n) - f(u_i^n)].$$

and we get

$$a_{i+1/2}^n = qf'(\xi_{i+1/2}^n) - 2qf'(\eta_2)\cdot\lambda \geqslant qf'(\xi_{i+1/2}^n) = q|f'(\xi_{i+1/2}^n)|.$$

.If $f'(\xi_{i+1/2}^n) = 0$, then $f(u_{i+1}^n) = f(u_i^n)$, and we use (51).

.If $(u_{i+1}^n - u_i^n)(f(u_{i+1/2}^n) - f(u_i^n)) < 0$, from (52), it follows that $a_{i+1/2}^n = q|f'(\xi_{i+1/2}^n)| = 0$.

.If $(u_{i+1}^n - u_i)(f(u_{i+1/2}^n) - f(u_i^n)) \geqslant 0$, then, from (51), $f(u_{i+1/2}^n) = f(u_i^n)$, and $a_{i+1/2}^n = 0$.

.If $f'(\xi_{i+1/2}^n) < 0$, and $f'(u_{i+1}^n) \geqslant 0$, we select for $u_{i+1/2}^n$ the root of f' the nearest of u_{i+1}^n in $\Gamma_{i+1/2}^n$, so that $f'(\eta_1) \geqslant 0$. Then from (52)

$$a_{i+1/2}^n(u_{i+1}^n - u_i^n) = q[f(u_i^n) - f(u_{i+1}^n)] + 2q[f(u_{i+1}^n) - f(u_{i+1/2}^n)];$$

and we get

$$a_{i+1/2}^n = -qf'(\xi_{i+1/2}^n) + 2(1-\lambda)qf'(\eta_1) \geqslant -qf'(\xi_{i+1/2}^n) = q|f'(\xi_{i+1/2}^n)|.$$

.If $f'(\xi_{i+1/2}^n) < 0$, and $f'(u_{i+1}^n) < 0$, then $u_{i+1/2}^n = u_{i+1}^n$, and from (52), $a_{i+1/2}^n = -qf'(\xi_{i+1/2}^n)$.

(22) is verified in each case, and thus Theorem 4 is proved.

Convergence to the solution in Kružkov's sense is also bound to the good choice of the root of f' in $\Gamma_{i+1/2}^n$ when $f'(\xi_{i+1/2}^n) = 0$ and

$$(u_{i+1}^n - u_i^n)(f(u_{i+1/2}^n) - f(u_i^n)) < 0,$$

as is shown by the numerical solution of the following problem

$$u_t + |u|(2u^2 - 1)u_x = 0; \qquad u(x, 0) = sg(x).$$

From (17) $u_i^0 = sg(i)$, therefore

$$u_{i+1/2}^0 = \begin{cases} 1 & \text{if } i \geqslant 1, \\ 0 \text{ or } 1/\sqrt{2} & \text{if } i = 0, \\ -1 & \text{if } i = -1 \text{ (from (51))}, \\ -1 & \text{if } i \leqslant -2. \end{cases}$$

By taking $u_{1/2}^0 = 0$, we have $u_{i+1/2}^n \in \{-1, 0, 1\}$; hence $u_i^1 = u_i^0$ for all i, and step by step, for all i, $u_i^{n+1} = u_i^n = \cdots = sg(i)$. The family $\{u_h\}$ converges to the stationary solution which does not satisfy the entropy condition. By taking $u_{1/2}^0 = 1/\sqrt{2}$, convergence to the solution in Kružkov's sense seems to become true. When f' has no root, convergence to the solution in Kružkov's sense is assured, since the Godounov scheme is reduced to the decentered scheme, studied below.

Lax-Wendroff Scheme. In [11], the Lax-Wendroff scheme is described in the following way, with $u_{i+1/2}^n = (u_i^n + u_{i+1}^n)/2$ for all $i \in \mathbf{Z}$,

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{q}{2}[f(u_{i+1}^n) - f(u_{i-1}^n)] \\ &\quad + \frac{q^2}{2}[f'(u_{i+1/2}^n)(f(u_{i+1}^n) - f(u_i^n)) - f'(u_{i-1/2}^n)(f(u_i^n) - f(u_{i-1}^n))]. \end{aligned} \tag{55}$$

We can give (55) the form (18) by taking

$$a_{i+1/2}^n = q^2 f'(u_{i+1/2}^n) f'(\xi_{i+1/2}^n). \tag{56}$$

From (35) this choice can make true the convergence to a weak solution which does not satisfy the entropy condition. On the other hand, we know that the Lax-Wendroff scheme is not stable in $L^\infty(\mathbf{R} \times]0, T[)$. To eliminate (to a certain extent) the oscillations near the shocks, Lax and Wendroff [6] propose adding to the second member of (55), a term of the form

$$\frac{q}{2}[b_{i+1/2}^n(u_{i+1}^n - u_i^n) - b_{i-1/2}^n(u_i^n - u_{i-1}^n)],$$

where $b_{i+1/2}^n$ is a function of u_{i+1}^n and u_i^n, vanishing if $u_i^n = u_{i+1}^n$, for example

$$b_{i+1/2}^n = \frac{1}{2}|f'(u_{i+1}^n) - f'(u_i^n)|/2.$$

This term does not alter the order of accuracy of the scheme (order 2) and efficiently reduces the oscillations near the shock, but does not necessarily assure the convergence to the solution in Kružkov's sense, as is shown by the numerical solution of the following problem

$$u_t + \sin(\pi u)u_x = 0; \qquad u(x, 0) = sg(x),$$

with the numerical initial condition

$$u_i^0 = \frac{1}{h}\int_{ih}^{(i+1)h} u_0(x)\,dx = \begin{cases} 1 & \text{if } i \geqslant 0, \\ -1 & \text{if } i < 0. \end{cases}$$

Then, for all i and j, $f'(u_i^n) = 0$, $f(u_j^n) = f(u_i^n)$. Therefore $u_i^1 = u_i^0$, and step by step $u_i^{n+1} = u_i^n$. The whole family u_h converges to the stationary solution $u(x, t) = sg(x)$, different from the solution in Kružkov's sense given by

$$u(x, t) = \begin{cases} 1 & \text{if } x > at, \\ \dfrac{1}{\pi} \text{ Arc } \sin(x/t) & \text{if } -at < x < at, \\ -1 & \text{if } x < -at, \text{ with } a = \dfrac{1}{\pi} \operatorname*{Sup}_{0 \leqslant u \leqslant 1} \left(\dfrac{1 + \cos \pi u}{1 - u}\right). \end{cases}$$

With the same initial condition, the numerical solution of Example 3 by the two-step Lax-Wendroff scheme defined by

$$\text{(57)} \qquad \begin{cases} u_{i+1/2}^{n+1/2} = (u_{i+1}^n + u_i^n)/2 - q(f(u_{i+1}^n) - f(u_i^n))/2, \\ u_i^{n+1} = u_i^n - q(f(u_{i+1/2}^n) - f(u_{i-1/2}^n)), \end{cases}$$

leads to the weak solution u_1, which is not the solution in Kružkov's sense. Other schemes of second order accuracy present the same drawbacks.

Interpretation. Let $h > 0$, $i \in \mathbf{Z}$, $n \leqslant N$; we consider a scheme of the form (18), and we suppose $u_{i+1}^n \neq u_i^n$. If condition (22) is verified, then

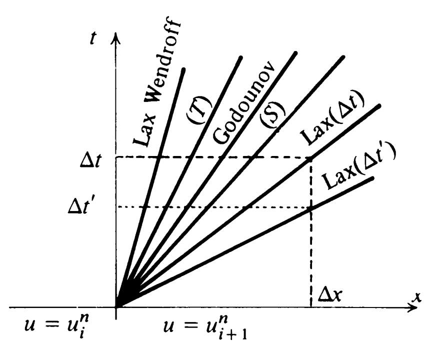
$$\text{(58)} \qquad a_{i+1/2}^n \frac{\Delta x}{\Delta t} \geqslant \left| \frac{f(u_{i+1}^n) - f(u_i^n)}{u_{i+1}^n - u_i^n} \right|.$$

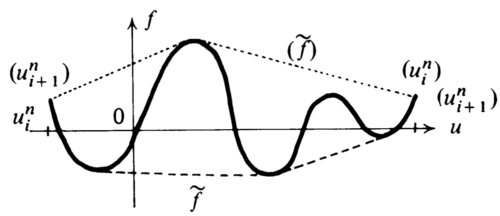
The quantity $a_{i+1/2}^n \Delta x/\Delta t$ can be compared to a speed; to assure convergence, it is sufficient, from (58), that this quantity be greater than or equal to the velocity module of a shock of intensity $|u_{i+1}^n - u_i^n|$. It is interesting, indeed, to compare (58) with the Rankine-Hugoniot equation (5). From that, we deduce, in particular, that the faster the shock, the greater the numerical viscosity. Condition (37) can be written as

$$\text{(59)} \qquad a_{i+1/2}^n \frac{\Delta x}{\Delta t} \geqslant s_{i+1/2}^n = \operatorname*{Sup}_{k \in \Gamma_{i+1/2}^n} \left\{ \operatorname{Max}\left(\frac{f(u_{i+1}^n) - f(k)}{u_{i+1}^n - k}, -\frac{f(u_i^n) - f(k)}{u_i^n - k} \right) \right\}.$$

which should be compared with (9), i.e. the expression of the velocity module of a shock satisfying the entropy condition. Thus $s_{i+1/2}^n$ expresses the velocity module of a shock, the intensity of which is equal to $|u_{i+1}^n - u_i^n|$, and which verifies the entropy condition. Such a shock is faster than a simple shock of intensity $|u_{i+1}^n -$

$u_i^n|$, and exists only if its speed also satisfies (5). Condition (59) means that the quantity $a_{i+1/2}^n \Delta x/\Delta t$ is superior or equal to the velocity module of such a shock. If the condition is satisfied, then we necessarily have convergence to the solution in Kružkov's sense. The shock which verifies the entropy condition is faster than the others, hence the necessity of a more important numerical viscosity. This relation between numerical viscosity and the velocity module of the shock is represented by a figure. By reducing Δt, the slope associated to the Lax scheme decreases (speed increases). For the other schemes, the slope remains unaltered. Let us underline the local nature of (58) and (59); the speed of a numerical shock is exact in general, since this shock is spread on several intervals I_i.





We can give another interpretation of $s_{i+1/2}^n$; it is the same as introducing convexity. We define on $\Gamma_{i+1/2}^n$ $\hat{f}$ as the convex hull of f if $u_i^n < u_{i+1}^n$, and as the concave hull of f if $u_i^n > u_{i+1}^n$. We have necessarily $\hat{f} \in C^1(\Gamma_{i+1/2}^n)$, and it follows that

$$s_{i+1/2}^n = \sup_{k \in \Gamma_{i+1/2}^n} |\hat{f}'(k)|. \tag{60}$$

The Case of Monotone f. In this case, the decentered scheme is applicable. It is written as

$$\begin{cases} \text{if } f \text{ is nondecreasing,} & u_i^{n+1} = u_i^n - q[f(u_i^n) - f(u_{i-1}^n)]\,, \\ \text{if } f \text{ is nonincreasing,} & u_i^{n+1} = u_i^n - q[f(u_{i+1}^n) - f(u_i^n)]\,. \end{cases} \tag{61}$$

This scheme is a particular case of the Godounov scheme but here the hypothesis of monotonicity assures convergence to the solution in Kružkov's sense, though (37) is not always verified.

THEOREM 5. *If the stability condition of Courant-Friedrichs-Lewy* (21) *is verified, then the family* $\{u_h\}_{h>0}$ *of approximate solutions, built by the decentered scheme when f is monotone, converges to the solution in Kružkov's sense of problem* (1), (2).

Proof. We can give (61) the form (18) in each case. As in Theorem 4, we obtain $a^n_{i+1/2} = q|f'(\xi^n_{i+1/2})|$. As stability condition (21) is verified, Theorem 1 allows us to conclude the convergence of a sequence $\{u_{h_m}\}_m$ to $u \in L^\infty(\mathbf{R} \times]0, T[)$. A formulation similar to (38) is also verified for the decentered scheme. Let $k \in \mathbf{R}$; if $M = |u_0|_{L^\infty(\mathbf{R})} < |k|$, we multiply (18) by $sg(M - k)$, hence (38) (see Theorem 2, Case 0), which may change into

$$
\begin{aligned}
|u_i^{n+1} - k| \leqslant |u_i^n - k| - \frac{q}{2}\,\{&sg(u^n_{i+1} - k)(f(u^n_{i+1}) - f(k)) \\
&- sg(u^n_{i-1} - k)(f(u^n_{i-1}) - f(k))\} \\
&+ b^n_{i+1/2}|u^n_{i+1} - u^n_i| - b^n_{i-1/2}|u^n_i - u^n_{i-1}|,
\end{aligned}
\tag{62}
$$

with

$$b^n_{i+1/2} = \frac{1}{2}\,a^n_{i+1/2}sg(u^n_{i+1} - u^n_i)sg\left(\frac{u^n_i + u^n_{i+1}}{2} - k\right).$$

When $|k| \leqslant M$, we can still state (62). By introducing $f(k)$ in (61), and by using (39), we get

$$
\begin{cases}
u_i^{n+1} - k = (u_i^n - k)(1 - qf'(k_i^n)) + (u^n_{i-1} - k)qf'(k^n_{i-1}) & \text{if } f' \geqslant 0,\\
u_i^{n+1} - k = (u_i^n - k)(1 + qf'(k_i^n)) + (u^n_{i+1} - k)(-qf'(k^n_{i+1})) & \text{if } f' \leqslant 0,
\end{cases}
$$

where all the coefficients are nonnegative. In both cases, if we take absolute values, we have

$$
\begin{aligned}
|u_i^{n+1} - k| \leqslant |u_i^n - k| - \frac{q}{2}\,\{&(f(u^n_{i+1}) - f(k))sg(u^n_{i+1} - k) \\
&- (f(u^n_{i-1}) - f(k))sg(u^n_{i-1} - k)\} \\
+ \frac{qs}{2}\,\{&(f(u^n_{i+1}) - f(k))sg(u^n_{i+1} - k) - (f(u^n_i) - f(k))sg(u^n_i - k)\} \\
- \frac{qs}{2}\,\{&(f(u^n_i) - f(k))sg(u^n_i - k) - (f(u^n_{i-1}) - f(k))sg(u^n_{i-1} - k)\},
\end{aligned}
$$

where $s = 1$, when $f' \geqslant 0$, $s = -1$ when $f' \leqslant 0$. We get (62) with a coefficient $b^n_{i+1/2}$ such that

$$b^n_{i+1/2}|u^n_{i+1} - u^n_i| = \frac{qs}{2}\{(f(u^n_{i+1}) - f(k))sg(u^n_{i+1} - k) - (f(u^n_i) - f(k))sg(u^n_i - k)\},$$

which is verified by

$$(63) \qquad b_{i+1/2}^n = \begin{cases} \dfrac{1}{2} a_{i+1/2}^n sg(u_{i+1}^n - u_i^n) sg\left(\dfrac{u_i^n + u_{i+1}^n}{2} - k\right) & \text{if } k \notin \Gamma_{i+1/2}^n, \\ \\ \dfrac{q}{2} [\lambda |f'(k_{i+1}^n)| - (1-\lambda)|f'(k_i^n)|] & \text{if } k \in \Gamma_{i+1/2}^n, \end{cases}$$

where $\lambda \in [0, 1]$ is given by $k = \lambda u_i^n + (1-\lambda) u_{i+1}^n$.

By introducing a function $\phi \in C_0^2(\mathbf{R} \times]0, T[)$, nonnegative, in (62), we get an expression similar to (42), from which we deduce (12) at the limit when h_m tends to zero, since $b_{i+1/2}^n$ remains bounded ($|b_{i+1/2}^n| \leqslant \frac{1}{2}$). With the same process as in the proof of Theroem 2 (part (b)), we state (13); and therefore, the whole family $\{u_h\}_{h>0}$ converges to the solution in Kružkov's sense of (1), (2), and thus Theorem 5 is proved.

When f is monotone, the derivation of f can be performed while preserving stability. In [12], Shampine and Thompson propose the following decentered scheme, when f is nondecreasing.

$$(64) \qquad u_i^{n+1} = u_i^n - qf'(u_i^n)(u_i^n - u_{i-1}^n).$$

If the stability condition of Courant-Friedrichs-Lewy (21) is satisfied then we can state the estimates (28), (29) and (30), and deduce the existence of a sequence $\{u_{h_m}\}$ convergent in $L_{\mathrm{loc}}^1(\mathbf{R} \times]0, T[)$ to a function $u \in L^\infty(\mathbf{R} \times]0, T[)$. But u is not necessarily a weak solution of (1), (2), as is shown by the following example. If $f(u) = u^2/2$ if $u > 0$ and 0 if $u \leqslant 0$, and $u_0(x) = -sg(x)$, it follows that $u_i^0 = -sg(i)$ and $f'(u_i^0) = 0$ if $i \geqslant 0$. Now if $i \leqslant -1$, $u_i^0 - u_{i-1}^0 = 0$, hence $f'(u_i^0)(u_i^0 - u_{i-1}^0) = 0$ for all $i \in \mathbf{Z}$. We have $u_i^1 = u_i^0$ for all i, and step by step $u_i^{n+1} = u_i^n$. The family $\{u_h\}$ converges to the stationary solution $u(x, t) = u_0(x)$, when h tends to zero, but u does not satisfy the Rankine-Hugoniot Eq. (5) which is a necessary condition for it to be a weak solution of problem (1), (2).

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