An Approximation for $\int_x^\infty e^{-t^{2/2}} t^p \, dt$, $x > 0$, $p$ real

By A. R. DiDonato

Abstract. A new approximation is given for $\int_x^\infty e^{-t^{2/2}} t^p \, dt$, $x > 0$, $p$ real, which extends an earlier approximation of Boyd's for $p = 0$.


$$g(x) = 4/[3x + \sqrt{x^2 + 8}]$$

as an approximation for

$$F(x) = \int_x^\infty Z(t) \, dt, \quad x > 0,$$

where

$$Z(x) = \exp(-x^2/2).$$

It can be shown $g(x) > F(x)$, and in fact

$$g(x) - F(x) = 2x^{-7} + O(x^{-9}), \quad (x \to \infty).$$

For $x > c \equiv (4 - \pi)/\sqrt{\pi(\pi - 2)} \approx .453$, $g(x)$ serves as a much better approximation to $F(x)$ than the well-known estimate

$$h(x) = 2/[x + \sqrt{x^2 + 8/\pi}], \quad [1, \text{p. 298}].$$

More specifically, it is easy to conclude that

$$F(x) < g(x) < h(x), \quad x > c,$$

as Table I below indicates.

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Our objective is to generalize Boyd's result to the function

$$F(p, x) = Y(p, x)/Z(p, x),$$

Received April 11, 1977.


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where

\( Y(p, x) = \int_x^\infty Z(p, t) \, dt, \)

\( Z(p, x) = x^p \exp(-x^2/2), \quad x > 0, \quad p \text{ real}. \)

The corresponding approximation for \( F(p, x) \) is given by

\( g(p, x) = 4x/\left[3(x^2 - p) + \sqrt{(x^2 - p)^2 + 8(x^2 + p)}\right], \)

which reduces to (1) for \( p = 0 \), and also is exact for \( p = 1 \), i.e., \( g(1, x) = F(1, x) \).

For fixed \( p \), it improves as \( x \) increases and, depending on the value of \( p \), it bounds \( F \) either from above or below for all \( x > x_m \). In particular,

\[
\begin{align*}
\text{(a)} & \quad p < 0, \quad x^2 > x_m^2 = -p \quad (\Rightarrow g(p, x) > F(p, x)), \\
\text{(b)} & \quad 0 < p \leq 1, \quad x^2 > x_m^2 = p + 2p^2/3 \quad (\Rightarrow g(p, x) \geq F(p, x))^*, \\
\text{(c)} & \quad p > 1, \quad x^2 > x_m^2 = p + 2p^2/3 \quad (\Rightarrow g(p, x) < F(p, x)).
\end{align*}
\]

By expanding (10) in powers of \( 1/x \) and subsequently taking the difference of the leading terms with those of the asymptotic series for \( F(p, x) \),

\[
F(p, x) \approx x \left[ \frac{1}{x^2} + \frac{(p - 1)(p - 3)}{x^4} + \cdots \right], \quad (x \to \infty),
\]

we find

\[
g(p, x) - F(p, x) \approx \frac{2(1 - p)}{x^7} + \frac{2(1 - p)(4p - 19)}{x^9} + O(x^{-11}), \quad (x \to \infty).**
\]

Table II is given to show the comparison between \( F(p, x) \) and \( g(p, x) \) for some selected values of \( p \) and \( x \). The asterisked \( x \) values are close approximates of \( x_m \) given in (11).

Before deriving (10), we note that an approximation, \( g_1(p, x) \), for \( F \) can also be obtained from the first two terms of the continued fraction expansion for the incomplete gamma function, namely

\[
g_1(p, x) = \frac{x^2 + 2}{x(x^2 + 3 - p)}, \quad [4, \text{p. 356}].
\]

The relationship between \( F \) and the incomplete gamma function is given below in (29).

From (12) and (14) one obtains

\[
g_1(p, x) - F(p, x) \approx 2(p - 1)(p - 3)/x^7 + O(x^{-9}), \quad (x \to \infty).
\]

A comparison with (13) shows that at large \( x \), \( g_1 \) is better than \( g \) for \( 2 < p < 4 \), but it is not as good otherwise, especially at large \( |p| \).

*For \( 0 < p < 1 \), computer results indicate \( 3p \) as a sharper estimate for \( x_m^2 \).

**Since the algebra is somewhat lengthy, the first two terms on the right-hand side of (12) and (13) were also computed by A. Morris, as well as three additional ones, using his algebraic computer program, "FLAP," [3].
Table II (see Table I for $p = 0$)

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Derivation of (10) and (11). We now derive (10) treating 11(a), (b) and (c) separately. The final results depend on the function $H(p, x)$ given below by (19) or (20).

Let

\begin{equation}
(16) \quad f(p, x) \equiv 1/F(p, x) = Z(p, x)/Y(p, x).
\end{equation}

For brevity, denote $f(p, x)$ by $f$, $\partial f/\partial x$ by $f'$, $\partial^2 f/\partial x^2$ by $f''$. Then

\begin{equation}
(17) \quad f' = f(f + p/x - x) = fu, \quad u \equiv f + p/x - x,
\end{equation}

\begin{equation}
(18) \quad f'' = fH, \quad H = H(p, x),
\end{equation}

where

\begin{equation}
(19) \quad H = 2f^2 + 3(p/x - x)f + (p/x - x)^2 - 1 - p/x^2,
\end{equation}

or

\begin{equation}
(20) \quad H = 2u^2 + (x - p/x)u - 1 - p/x^2 = u' + u^2.
\end{equation}

We shall use the following properties of $H$, which are easily found:
\[
\lim_{x \to 0^+} H(p, x) = \begin{cases} 
\infty, & p < 0, \\
(4-\pi)/\pi, & p = 0, \\
-\infty, & 0 < p < 1, \\
\infty, & p > 1.
\end{cases}
\]

Also
\[
H(p, x) \approx 2(1-p)/x^4, \quad (x \to \infty),
\]
which follows after some tedious algebra from (12) and (19).

For \( p < 0 \), we have
\[
u = f + p/x - x > 0, \quad x > s = \sqrt{-p}.
\]
Indeed, let \( S = e + y = z - (x - p/x)y \), so that \( \partial S/\partial x \leq 0 \) for \( x^2 \geq -p \), and \( S \leq z/x^2 \) \((x \to \infty)\). Hence \( S > 0 \) for \( x^2 > -p \), and since \( y \) is always positive (23) follows.

From this result, with (21) and (22), we have \( H > 0 \) for \( x \in [s, \infty) \). In fact, if this were not the case, there would exist a point \( \xi \in (s, \infty) \) for which \( H(p, \xi) = 0 \), \( H'(p, \xi) > 0 \). But this is impossible since
\[
H'(p, x) = (f + 2u)H(p, x) - 2u^3 + 2p/x^3,
\]
is negative if \( H = 0, u > 0, \) and \( p < 0 \).

Thus factoring \( H \), as given in (19),
\[
H = (f - \eta_+)(f - \eta_-) > 0, \quad x^2 \geq -p, \quad p < 0,
\]
we get (10) with 11(a) from \( f - \eta_+ > 0 \), where
\[
\eta_\pm(p, x) = \left[3(x - p/x) \pm \sqrt{(x - p/x)^2 + 8(1 + p/x^2)}\right]/4.
\]

Now consider the case \( p > 1 \). From (21) and (22) we know \( H \) has at least one positive zero such that \( H'(p, x_0) \leq 0 \), where \( x_0 \) denotes the largest such zero. Moreover, if \( z \) denotes the largest zero of \( H \) with \( H'(p, z) > 0 \), then \( z < x_0 \). Thus \( H \leq 0 \) for all \( x > x_0 \). In order to get an estimate, \( x_m \), of \( x_0 \) we have from (24) and \( H'(p, x_0) \leq 0 \), that
\[
u(p, x_0) > p^{1/3}/x_0, \quad p > 1.
\]
Inequality 11(c) now follows directly by using (27) and \( f(p, x_0) = \eta_+(p, x_0) \) in the expression for \( u \) given in (17).

When \( 0 < p < 1 \), the analysis used to obtain 11(b) is similar to that for \( p > 1 \). First, it is shown (27) holds with the inequality reversed. Then proceeding as above, one obtains 11(b) with \( H(p, x) > 0 \) for all \( x > x_m > x_0 \). The details are omitted.

**Relation of \( F(p, x) \) to the Incomplete Gamma Function.** The quantity \( F(p, x) \) can be related to the normalized incomplete gamma function. Let
\[
r = t^2/2, \quad y = x^2/2,
\]
so that
(29) \[ F(p, x) = \frac{1}{\sqrt{2}} e^{x} y^{-p/2} \Gamma\left(\frac{p + 1}{2}, y\right), \]
where

(30) \[ \Gamma(a, z) = \int_{z}^{\infty} e^{-y} y^{-a-1} \, dy, \quad z > 0, a \text{ real.} \]

Thus, we have from (10) and (11)

(31) \[ \Gamma(a, y) = \left( \frac{4e^{-y} y^a}{3[y - a + \frac{1}{2}] + \sqrt{(y - a + \frac{1}{2})^2 + 4(y + a - \frac{1}{2})}} \right), \]

with

\[
\begin{align*}
(a) \quad & a \leq \frac{1}{2}, \quad y \geq y_m \equiv \frac{1}{2} - a, \quad (\Rightarrow g(2a - 1, \sqrt{2}y) > \Gamma(a, y)), \\
(b) \quad & \frac{1}{2} < a \leq 1, \quad y \geq y_m \equiv a - \frac{1}{2} + (2a - 1)^{2/3}, \quad (\Rightarrow g(2a - 1, \sqrt{2}y) \geq \Gamma(a, y)), \\
(c) \quad & a > 1, \quad y \geq y_m \equiv a - \frac{1}{2} + (2a - 1)^{2/3}, \quad (\Rightarrow g(2a - 1, \sqrt{2}y) \leq \Gamma(a, y)).
\end{align*}
\]

An Application. The function \( g(p, x) \) is particularly good for giving quick estimates when \( p < 0 \) and \( F(p, x) \) cannot be evaluated by the usual recurrence relationship on \( p \). In addition, \( g \) can often be used to establish properties of \( F \). For example, a result not obtained easily by direct means, is to find, for \( p < 0 \), a good estimate to \( x = X \), where \( F(p, X) > F(p, x) \) for all \( x > 0 \). In fact, by using (10) and noting that \( F'(p, X) = 0 \) requires \( F(p, X) = X/(X^2 - p) \), we find

(33) \[ g(p, X) - F(p, X) \approx -X(X^2 + p)/(X^2 - p)^3. \]

Therefore, with \( X^2 \approx -p \),

(34) \[ F(p, X) \approx F(p, \sqrt{-p}) \approx g(p, \sqrt{-p}) = 1/(2\sqrt{-p}). \]

Clearly (34) also implies that \( F(p, X) \leq 1/(2\sqrt{-p}) \). Thus, if \( p = -100 \), then \( X = 9.950374, \sqrt{-p} = 10, F(-100, X) = 0.04999938, 1/(2\sqrt{-p}) = .05, g(p, X) = .05000063 \). Even for the case of \( p = -4 \), where one does not expect the right side of (33) to hold, we find \( X = 1.791507, \sqrt{-p} = 2, F(-4, X) = .2484926, 1/(2\sqrt{-p}) = .250, g(-4, X) = .2524567 \).

Acknowledgment. My thanks to Mr. Richard K. Hageman for his programming help, and the computation of Tables I and II.

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