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On the Observed Rate of Convergence of an Iterative Method Applied to a Model Elliptic Difference Equation

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Abstract. A proof is given of the fact that the rate of convergence of a multiple grid type of algorithm is $O(h^{1/2})$ in the case of a model elliptic difference equation.

1. This note provides a theoretical explanation of the $O(h^{1/2})$ rate of convergence observed in the application of a multiple grid method to a model finite difference Poisson problem. The method in question was proposed in [2] and some numerical results were given from which the $O(h^{1/2})$ figure was inferred. This method consists of applying a second degree acceleration technique to the iterates from a linear stationary iterative method of the first degree. Only the latter involves anything new—it is the multiple grid part of the algorithm—and is all that will be considered here. More specifically, once the spectral properties of the iterating matrix of the first degree process are established, the properties of the composite algorithm can be deduced by standard techniques. These were discussed in [2] and well-known books and papers [1], [3], [4] may be consulted for more details. Use of these techniques will be made in Section 4. In Sections 2 and 3 the necessary theoretical preliminaries are developed.

2. The problem considered in [2] was the solution of the partial difference equation

$$L_h u_{ij}^h = f_{ij}^h, \qquad i, j = 1, 2, \ldots, n, \tag{2.1}$$

on the $(n + 2) \times (n + 2)$ square grid Ω_h (with grid boundary $\partial\Omega_h$) of side length π and grid points of the form

$$P_{ij} = (ih, jh), \qquad i, j = 0, 1, \ldots, n + 1; \qquad (n + 1)h = \pi.$$

u_{ij}^h is the sought for grid function evaluated at P_{ij} and u_{ij}^h vanishes on $\partial\Omega_h$. f_{ij}^h is a given grid function and the operator L_h is the discrete Laplacian operator defined below.

For grid functions v^h defined on $\overline{\Omega}_h$ the following operators are required:

$$\Delta_1^h v_{ij}^h = h^{-1}(v_{i+1j}^h - v_{ij}^h), \qquad 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n + 1,$$

$$\nabla_1^h v_{ij}^h = h^{-1}(v_{ij}^h - v_{i-1j}^h), \qquad 1 \leqslant i \leqslant n + 1, 0 \leqslant j \leqslant n + 1,$$

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with corresponding definitions for Δ_2^h and ∇_2^h the forward and backward difference operators in the y direction. In this notation the formula for summation by parts becomes

$$\sum_{i=0}^{n} v_{ij}^h \Delta_1^h w_{ij}^h = h^{-1} v_{ij}^h w_{ij}^h \Big|_{i=0}^{n+1} - \sum_{i=1}^{n+1} w_{ij}^h \nabla_1^h v_{ij}^h \tag{2.2}$$

and the discrete Laplacian L_h introduced above is defined by

$$L_h v_{ij}^h = -\Delta_1^h \nabla_1^h v_{ij}^h - \Delta_2^h \nabla_2^h v_{ij}^h. \tag{2.3}$$

A notational convention that will be used from now on is that undefined difference quotients such as $\nabla_1^h u_{0j}^h$ are defined to be zero. With this understanding it follows from (2.3) using (2.2) that if u_{ij}^h vanishes on $\partial\Omega_h$, then

$$\sum_{i,j=0}^{n+1} u_{ij}^h L_h u_{ij}^h = \sum_{i,j=0}^{n+1} [(\nabla_1^h u_{ij}^h)^2 + (\nabla_2^h u_{ij}^h)^2], \tag{2.4}$$

which of course is an analogue of a result widely used in the continuous case. Another definition is needed; if v^h and w^h are grid functions defined on some subset $\overline{\Gamma}_h$ of $\overline{\Omega}_h$, then

$$(v^h, w^h)_{\overline{\Gamma}_h} = h^2 \sum_{i,j \in \overline{\Gamma}_h} v_{ij}^h w_{ij}^h, \tag{2.5}$$

and also

$$|v^h|_{\overline{\Gamma}_h}^2 = (v^h, v^h)_{\overline{\Gamma}_h}^2 .$$

The summation indices on (2.5) are to be taken over the set of points in $\overline{\Gamma}_h$. (2.4) then can be written as

$$(u^h, L_h u^h)_{\overline{\Omega}_h} = |\nabla_1^h u^h|_{\overline{\Omega}_h}^2 + |\nabla_2^h u^h|_{\overline{\Omega}_h}^2 = |\Delta_1^h u^h|_{\overline{\Omega}_h}^2 + |\Delta_2^h u^h|_{\overline{\Omega}_h}^2 . \tag{2.6}$$

The notational convention introduced above is in action on both of these equations.

The lemma which follows is a discrete analogue of Poincaré's inequality, essential to later arguments. The proof is little more than a discrete copy of a proof well known in the continuous case but will be given for completeness and because it is short.

LEMMA 2.1 (POINCARÉ). *Let $\overline{\Gamma}_h$ be any $(n+2) \times (n+2)$ square grid with spacing $h = A/(n+1)$, and let v^h be any grid function defined on $\overline{\Gamma}_h$. Then*

$$|v^h|_{\overline{\Gamma}_h}^2 \leqslant A^2 \left(\frac{n+2}{n+1}\right) [|\Delta_1^h v^h|_{\overline{\Gamma}_h}^2 + |\Delta_2^h v^h|_{\overline{\Gamma}_h}^2] + (n+2)^{-2} \left[\sum_{i,j \in \overline{\Gamma}_h} v_{ij}^h\right]^2 .$$

Proof. Note. h is used as a symbol here to avoid proliferation of notations. It is not related to the h used earlier.

Name the grid points P_{ij}, $i, j = 0, 1, \ldots, n+1$, in the usual way. It is clear that

$$v_{ij}^h - v_{kj}^h = \operatorname{sgn}(i-k)h \sum_{\alpha=\min(i,k)}^{\max(i,k)-1} \Delta_1^h v_{\alpha j}^h$$

and, consequently, that

$$v_{ij}^h - v_{kl}^h = \operatorname{sgn}(i-k)h \sum_{\alpha=\min(i,k)}^{\max(i,k)-1} \Delta_1^h v_{\alpha j}^h + \operatorname{sgn}(j-l)h \sum_{\beta=\min(j,l)}^{\max(j,l)-1} \Delta_2^h v_{k\beta}^h.$$

Squaring both sides, using Cauchy's inequality twice and increasing the summation indices yields

$$(v_{ij}^h)^2 + (v_{kl}^h)^2 - 2(v_{ij}^h v_{kl}^h) \leqslant 2h^2(n+1)\left[\sum_{\alpha=0}^{n} (\Delta_1^h v_{\alpha j}^h)^2 + \sum_{\beta=0}^{n} (\Delta_2^h v_{k\beta}^h)^2\right].$$

Summing for $i, j = 0, 1, \ldots, n+1$, then for $k, l = 0, 1, \ldots, n+1$, gives

$$(n+2)^2 |v^h|_{\bar{\Gamma}_h}^2 + (n+2)^2 |v^h|_{\bar{\Gamma}_h}^2 - 2\left(\sum_{i,j\in\bar{\Gamma}_h} v_{ij}^h\right)^2$$

$$\leqslant 2h^2(n+1)(n+2)^3 [|\Delta_1^h v^h|_{\bar{\Gamma}_h}^2 + |\Delta_2^h v^h|_{\bar{\Gamma}_h}^2],$$

which is equivalent to the stated result.

In the next section some facts about the iterative method under discussion are collected together.

3. To introduce the algorithm suggested in [2], first of all, the difference equation (2.1) must be put into matrix form. To do this, an arbitrary fixed order is assigned to the grid points of $\bar{\Omega}_h$ and the equation (2.1) written for the points of Ω_h in the order of their listing. This is the usual procedure. The result of it is a system of linear equations, N of them, which will be denoted by $A_h u^h = f^h$, where A_h is an $N \times N$ positive definite matrix with $4/h^2$ in each diagonal position, and the meaning of the vectors u^h and f^h is the obvious one.

No confusion will arise with the earlier use of these symbols. Secondly, the internal grid points are partitioned into subsets as follows; take $N = n^2$ for some positive integer n so that there are n^4 points in Ω_h, and divide them into n^2 sets of n^2 points by drawing $n^2 - 1$ lines equally spaced and parallel to the x axis and again for the y axis. List the subsets in some order and denote them by $\Omega_h^{(i)}$, $i = 1, 2, \ldots, n^2$. For any vector v^h defined at the points of Ω_h', a "contracted" vector $\bar{v}^h$ is defined by:

$$(\bar{v}^h)_i = \frac{1}{n} \sum_{k\in\Omega_h^{(i)}} (v^h)_k, \qquad i = 1, 2, \ldots, n^2. \tag{3.1}$$

This transformation may be represented by a matrix $(E_h)^T$, where E_h has dimensions $n^4 \times n^2$ and the ith column of E_h, associated with $\Omega_h^{(i)}$, has $1/n$ in those positions with numbers corresponding to grid points in $\Omega_h^{(i)}$ and zeros in the other positions. The matrix $E_h^T E_h$ will then be the unit matrix of order $n^2 \times n^2$. With these notations, the matrix

$$(3.2) \qquad (I - E_h(E_h^T A_h E_h)^{-1} E_h^T A_h)\left(I - \frac{h^2}{4} A_h\right)$$

was found in [2] to be the iterating matrix of the first degree method referred to in Section 1. The relevant spectral properties of (3.2) are deduced below.

Let

$$(3.3) \qquad \Pi_h = (I - E_h(E_h^T A_h E_h)^{-1} E_h^T A_h), \quad B_h = \left(I - \frac{h^2}{4} A_h\right).$$

LEMMA 3.1. *The eigenvalues of* $\Pi_h B_h$ *are real.*

Proof. Let $S_h^2 = A_h$, with S_h positive definite. Then $S_h \Pi_h S_h^{-1}$ is similar to Π_h and is an orthogonal projection matrix. Also $S_h B_h = B_h S_h$ and so $\Pi_h B_h$ is similar to

$$S_h \Pi_h S_h^{-1} B_h = (S_h \Pi_h S_h^{-1})^2 B_h.$$

The latter has the same eigenvalues as the symmetric matrix

$$(S_h \Pi_h S_h^{-1}) B_h (S_h \Pi_h S_h^{-1})$$

and the lemma follows.

Let $\mathcal{E}_h$ denote span(E_h) and let $P_h = E_h(E_h^T E_h)^{-1} E_h^T$ denote the orthogonal projector onto $\mathcal{E}_h$. Put also $Q_h = I - P_h$.

LEMMA 3.2. *Let* $w^h \in \mathcal{E}_h^\perp$. *Then the corresponding grid function* w_{ij}^h *satisfies*

$$(3.4) \qquad \sum_{i,j \in \Omega_h^{(k)}} \mathrm{w}_{ij} = 0, \qquad k = 1, 2, \ldots, n^2.$$

Proof. w^h necessarily satisfies

$$w^h = (I - E_h(E_h^T E_h)^{-1} E_h^T) s^h \equiv (I - E_h E_h^T) s^h$$

for some vector s^h. The vector on the right consists of a vector s^h minus a vector which is constant in each $\Omega_h^{(k)}$ and equal to the average value of s^h over $\Omega_h^{(k)}$. So the average of w^h is zero over each $\Omega_h^{(i)}$ and this is equivalent to (3.4).

The eigenvalues of $\Pi_h B_h$, γ, satisfy for some real vectors ζ^h,

$$(3.5) \qquad \Pi_h B_h \zeta^h = \gamma \zeta^h.$$

Multiplying on the left of (3.5) by Q_h and using $Q_h E_h = 0$ gives

$$(3.6) \qquad Q_h B_h \zeta^h = \gamma Q_h \zeta^h$$

and multiplying on the left of (3.5) by $P_h A_h$ gives after a little simplification when $\gamma \neq 0$

$$Q_h A_h \zeta^h = A_h \zeta^h. \tag{3.7}$$

Put $\delta = 1 - \gamma$, $\delta \neq 1$. Substituting from (3.3) for B_h in (3.6), it follows that

$$\frac{h^2}{4} Q_h A_h \zeta^h = \delta Q_h \zeta^h = \frac{h^2}{4} A_h \zeta^h$$

by (3.7).

Multiplying on the left by A_h^{-1} and taking the (vector) inner product of both sides with $Q_h \zeta^h$, there results the equation for δ,

$$= \frac{h^2}{4} \frac{(Q_h \zeta^h, Q_h \zeta^h)}{(Q_h \zeta^h, A_h^{-1} Q_h \zeta^h)}, \tag{3.8}$$

where the property $Q_h^T Q_h \equiv Q_h^2 = Q_h$ has been used. (3.8) will be used to bound γ above and below. This will be done in Section 4.

4. If u_{ij}^h and v_{ij}^h are grid functions vanishing on $\partial \Omega_h$, the following relations may be (trivially) verified

$$(u^h, v^h)_{\overline{\Omega}_h} = h^2 (u^h, v^h), \qquad (u^h, L_h v^h)_{\overline{\Omega}_h} = h^2 (u^h, A_h v^h), \tag{4.1}$$

where the quantities in parentheses on the right are vector inner products. To obtain a lower bound for γ the simple inequality

$$(u^h, u^h)^2 \leqslant (u^h, A_h u^h)(u^h, A_h^{-1} u^h)$$

actually valid for any positive definite matrix, may be used. Thus,

$$1 - \gamma \leqslant \frac{h^2}{4} \frac{(u^h, A_h u^h)}{(u^h, u^h)} < 2,$$

the last coming, e.g. from Gershgorin's theorem (strengthened as in [3]). So then

$$\gamma > -1. \tag{4.2}$$

By using other arguments, unnecessary here, it may be shown that $\gamma > -1 + Kh^2$. This is well known. To obtain the upper bound is less simple.

To begin with, define w^h by $A_h w^h = Q_h \zeta^h$, where $Q_h \zeta^h$ appears in (3.8). Then (3.8) becomes

$$\delta = \frac{h^2}{4} \frac{(A_h w^h, A_h w^h)}{(w^h, A_h w^h)} \geqslant \min_{A_h w^h \in E_h^{\perp}} \frac{h^2}{4} \frac{(A_h w^h, A_h w^h)}{(w^h, A_h w^h)}.$$

This minimum value is estimated (actually sharply to within a multiplicative constant) in the following theorem.

THEOREM 4.1. *Let s^h be any vector such that $A_h s^h \in E_h^{\perp}$ and let s_{ij}^h be the* *corresponding grid function vanishing on $\partial\Omega_h$. Then*

$$(L_h s^h, L_h s^h)_{\overline{\Omega}_h} \geqslant (\Pi h)^{-1}(s^h, L_h s^h)_{\overline{\Omega}_h}.$$

Proof. Let $A_h s^h = w^h \in E_h^\perp$, and let w_{ij}^h be the corresponding grid function vanishing on Ω_h. Then $(s^h, A_h s^h) = (s^h, w^h)$. By (2.6), (4.1) and the last equation,

$$(s^h, L_h s^h)_{\overline{\Omega}_h} = |\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h} |\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h}$$

$$= (s^h, w^h)_{\overline{\Omega}_h} = h^2 \sum_k \sum_{\overline{\Omega}_h^{(k)}} w_{ij}^h s_{ij}^h.$$

Let $\overline{s}_k^h$ denote the mean of s_{ij}^h over $\overline{\Omega}_h^{(k)}$, so that

$$\overline{s}_k^h = \frac{1}{|\overline{\Omega}_h^{(k)}|} \sum_{i,j \in \overline{\Omega}_h^{(k)}} s_{ij}^h.$$

Then since $w^h \in E_h^\perp$ and by Lemma 3.2, it follows that

$$|\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h} = h^2 \sum_k \sum_{\overline{\Omega}_h^{(k)}} w_{ij}^h (s_{ij}^h - \overline{s}_k^h). \tag{4.3}$$

By Lemma 2.1, with $A^2 = [(n-1)\pi/(n^2+1)]^2$, and making use of the vanishing over $\overline{\Omega}_h^{(k)}$ of $\Sigma(s_{ij}^h - \overline{s}_k^h)$, one has

$$|s^h - \overline{s}|^2_{\overline{\Omega}_h^{(k)}} \leqslant \left(\frac{n}{n-1}\right) A^2 (|\Delta_1^h (s^h - \overline{s}^h)|^2_{\overline{\Omega}_h^{(k)}} + |\Delta_2^h (s^h - \overline{s}^h)|^2_{\overline{\Omega}_h^{(k)}}); \tag{4.4}$$

and furthermore,

$$|\Delta_i^h (s^h - \overline{s}^h)|^2_{\overline{\Omega}_h^{(k)}} = |\Delta_i^h s^h|^2_{\overline{\Omega}_h^{(k)}}, \qquad i = 1, 2. \tag{4.5}$$

Applying Cauchy's inequality to (4.3) and using (4.4) and (4.5),

$$|\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h} \leqslant h^2 \sum_k \left(\sum_{\overline{\Omega}_h^{(k)}} (w_{ij}^h)^2 \right)^{1/2} \left(\sum_{\overline{\Omega}_h^{(k)}} (s_{ij}^h - \overline{s}_k^h)^2 \right)^{1/2}$$

$$= \sum_k |w^h|_{\overline{\Omega}_h^{(k)}} |s^h - \overline{s}^h|_{\overline{\Omega}_h^{(k)}}$$

$$\leqslant \left(\frac{n+1}{n}\right)^{1/2} A \sum_k |w^h|_{\overline{\Omega}_h^{(k)}} [|\Delta_1^h s^h|^2_{\overline{\Omega}_h^{(k)}} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h^{(k)}}]^{1/2}$$

$$\leqslant \left(\frac{n+1}{n}\right)^{1/2} A |w^h|_{\overline{\Omega}_h} [|\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h}]^{1/2}$$

by Cauchy's inequality again, and using the fact that

$$\left(\sum_k |\Delta_1^h s^h|^2_{\overline{\Omega}_h^{(k)}} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h^{(k)}}\right)^{1/2} \leqslant (|\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h})^{1/2},$$

this reduces to

$$|\Delta_1^h s^h|^2_{\overline{\Omega}_h} + |\Delta_2^h s^h|^2_{\overline{\Omega}_h} \leqslant \left(\frac{n}{n-1}\right) A^2 |w^h|^2_{\overline{\Omega}_h} \leqslant \pi h |w^h|^2_{\overline{\Omega}_h};$$

and since $(w^h, w^h)_{\overline{\Omega}_h} = (L_h s^h, L_h s^h)^2_{\overline{\Omega}_h}$, the result is proved.

It now follows, making use of the expression for δ given before the statement of Theorem 4.1, that

$$\delta \geqslant \frac{h^2}{4} \cdot \frac{1}{\pi h} = \frac{h}{4\pi}$$

and finally that

$$\gamma \leqslant 1 - \frac{h}{4\pi}. \tag{4.6}$$

Thus, all the eigenvalues of $\Pi_h B_h$ are contained within the interval $(-1, 1 - h/4\pi)$. It will now be shown that the composite iteration consisting of applying the second degree method to the iterates from $\Pi_h B_h$ converges at the rate $O(h^{1/2})$. It is known that to do this, the quantity

$$\bar{\mu} = \frac{[1 - (-1)] - [(1 - (1 - h/4\pi))]}{[1 - (-1)] + [(1 - (1 - h/4\pi))]}$$

and its square are formed. The effective rate of convergence is then given by

$$\ln \left(\frac{1 - \sqrt{(1 - \bar{\mu}^2)}}{1 + \sqrt{(1 - \bar{\mu}^2)}}\right)^{1/2}.$$

Expanding in series and neglecting all but the smallest powers of h gives for this latter the required $O(h^{1/2})$ figure. The quantity called in [2] the effective spectral radius is similarly computed and turns out to be $1 - h^{1/2}/\sqrt{2\pi}$. The empirically observed value was $1 - 2h^{1/2}/\sqrt{\pi}$.

Finally, it seems reasonable that the results found above could be adapted to generalizing the rate of convergence result to other second order elliptic problems.

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