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Computation of the Bivariate Normal Integral

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Abstract. This paper presents a simple and efficient computation for the bivariate normal integral based on direct computation of the double integral by the Gauss quadrature method.

1. **Introduction.** The probability distribution of the normalized Normal Distribution is: [1]

$$\begin{aligned} &\Phi(h, k, \rho) = \Pr\{(x_1 < h) \cap (x_2 < k)\}, \\ &\Phi(h, k, \rho) = (2\pi\sqrt{1-\rho^2})^{-1} \int_{-\infty}^{h} \int_{-\infty}^{k} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right] dx_1\, dx_2. \end{aligned} \tag{1}$$

Substitute:

$$u_1 = \frac{h - x_1}{[2(1-\rho^2)]^{1/2}}; \quad u_2 = \frac{k - x_2}{[2(1-\rho^2)]^{1/2}}. \tag{2}$$

Define:

$$h_1 = \frac{h}{[2(1-\rho^2)]^{1/2}}; \quad k_1 = \frac{k}{[2(1-\rho^2)]^{1/2}}. \tag{3}$$

Then:

$$\begin{aligned} \Phi(h, k, \rho) = \frac{(1-\rho^2)^{1/2}}{\pi} \int_0^\infty \int_0^\infty &\exp[-u_1^2]\exp[-u_2^2] \\ &\exp[h_1(2u_1 - h_1) + k_1(2u_2 - k_1) + 2\rho(u_1 - h_1)(u_2 - k_1)]\, du_1\, du_2. \end{aligned} \tag{4}$$

By Gauss quadrature [2]:

$$\Phi(h, k, \rho) \doteq \frac{(1-\rho^2)^{1/2}}{\pi} \sum_{i,j=1}^{k} A_i A_j f(x_i, x_j), \tag{5}$$

where

$$f(x, y) = \exp[h_1(2x - h_1) + k_1(2y - k_1) + 2\rho(x - h_1)(y - k_1)]. \tag{6}$$

The values of A_i, x_i for $k = 2, \ldots, 15$, can be found in [3]. If $h, k, \rho \leqslant 0$, then $0 < f(x, y) \leqslant 1$, and the error in (5) is relatively small. We will make use of the following formulae in order to calculate the double integral for $h, k, \rho \leqslant 0$.

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2. **The Method.** The following formulae can be found in [1]:

$$\Phi(h, k, \rho) = \phi(h) + \phi(k) - 1 + \Phi(-h, -k, \rho), \tag{7}$$

$$\Phi(h, k, \rho) = \phi(k) - \Phi(-h, k, -\rho), \tag{8}$$

$$\Phi(h, k, \rho) = \phi(h) - \Phi(h, -k, -\rho), \tag{9}$$

where

$$\phi(h) = \frac{1}{2\pi}\int_{-\infty}^{h} \exp\left[-\frac{x^2}{2}\right] dx. \tag{10}$$

For $h, k \neq 0$

$$\Phi(h, k, \rho) = \Phi(h, 0, \rho(h, k)) + \Phi(k, 0, \rho(k, h)) - \delta_{hk}, \tag{11}$$

where

$$\rho(h, k) = \frac{(\rho h - k)\mathrm{Sgn}(h)}{\sqrt{h^2 - 2\rho hk + k^2}}, \quad \delta_{hk} = \frac{1 + \mathrm{Sgn}(h) \cdot \mathrm{Sgn}(k)}{4} \tag{12}$$

and

$$\mathrm{Sgn}(x) = \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0 \end{cases}.$$

Algorithm. If $h \cdot k \cdot \rho \leqslant 0$, then do one of the following:

(a) If $h \leqslant 0, k \leqslant 0, \rho \leqslant 0$ compute directly.

(b) If $h \leqslant 0, k \geqslant 0, \rho \geqslant 0$ use (9).

(c) If $h \geqslant 0, k \leqslant 0, \rho \geqslant 0$ use (8).

(d) If $h \geqslant 0, k \geqslant 0, \rho \leqslant 0$ use (7).

If $h \cdot k \cdot \rho > 0$, use (11). Note that every computation of Φ will now satisfy $h \cdot k \cdot \rho = 0$, (since the new k equals 0).

3. **Results.** An advantage of this method is that for every ρ (even close to 1) there is no convergence problem. In Table 1 we present results of the average run time and maximum-error for various values of k in (5).

Low values of the exponent in (6) cause $f(x, y)$ to vanish. To save computational effort, if the exponent is lower than the values in Column 4, Table 1, we assume that $f(x, y)$ is zero. The values in the table have been set such that maximum-error remains the same to two significant digits. In Column 5 we present the reduced run time. In order to compare with existing results [4], [5] we take $k = 5$. Note here that in $k = 3, 4, 5$ a further reduction of 0.7 m.s. can be achieved by using the approximation in [6] for the error function instead of using the function erf.

By a regression on Column 3 we can deduce that computations outside the double integration are the same for every k and require approximately 1.6 m.s., so for $k = 5$ and the approximated error function the average computation time is 2.2 m.s. The double integration requires an average 1.3 m.s.

TABLE 1. *Results*

k	maximum error	run time (10^{-3}sec)	limit of exponent	reduced run time
3	1.1×10^{-4}	3.0	−8	1.8
4	8.1×10^{-6}	4.1	−10	2.2
5	5.5×10^{-7}	5.5	−12	2.9
6	3.8×10^{-8}	7.2	−15	3.6
7	3.0×10^{-9}	9.2	−17	4.6
8	2.2×10^{-10}	11.5	−20	6.0
9	1.5×10^{-11}	14.2	−22	7.5
10	1.1×10^{-12}	17.0	−25	9.4

All time data are based on a CDC/6400.

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