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A Double Shooting Scheme for Certain Unstable and Singular Boundary Value Problems

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Abstract. A scheme is presented to obtain the unique bounded solution for an exponentially unstable linear system. The scheme consists of choosing random data at large initial values and integrating forwards and backwards until accurate regular boundary values are obtained. Proofs of convergence are given for the case that the homogeneous equation has an exponential dichotomy. Applications to other types of problems are discussed and numerical results are presented.

I. Introduction. We consider here linear inhomogeneous equations

$$\dot{y} = A(t)y + f, \tag{1.1}$$

with $A(t)$ an $n \times n$ matrix and y and f n-vectors, defined on some open interval I' of the real axis. We suppose that (1.1) has the property that whenever $f \in L_\infty(I')$, there exists a unique solution $y_\infty \in L_\infty(I')$. Massera and Schäffer [4] have studied such systems extensively and call such a property (L_∞, L_∞) admissibility. When $I' = (-\infty, \infty)$, it is possible to give a complete characterization of such systems and for simplicity we restrict ourselves to this case, although numerical results will be given for a system defined on $(0, \infty)$.

For such equations it follows that all nontrivial solutions to the homogeneous equation

$$\dot{y} = Ay \tag{1.2}$$

are unbounded as $t \longrightarrow \infty$ or $-\infty$. If all solutions are unbounded as $t \longrightarrow -\infty$ and in fact decay as $t \longrightarrow +\infty$ (for example if A is a constant matrix whose eigenvalues have negative real part), then the numerical computation of y_∞ presents no difficulties as the system (1.1) is stable in the forward direction (see Bayliss [1]). We consider here the more general case where (1.2) has solutions which grow as $t \longrightarrow \pm\infty$ and specifically where the homogeneous equation has an exponential dichotomy (see Massera and Schäffer).

An equation of the form (1.2) is said to have an exponential dichotomy if there exist projections P_1, $P_2 = I - P_1$, independent of t, such that if $Y(t)$ is the fundamental matrix solution to (1.2) ($Y(0) = I$) we have for some constants K and α

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$$\|Y(t)P_1Y^{-1}(s)\| \leqslant Ke^{-\alpha(t-s)}, \qquad t \geqslant s,$$
$$\|Y(t)P_2Y^{-1}(s)\| \leqslant Ke^{-\alpha(s-t)}, \qquad s \geqslant t. \tag{1.3}$$

(The definition of admissibility as originally given by Massera and Schäffer does not require uniqueness of the bounded solution y_∞. Also the definition of exponential dichotomy given here is more restrictive than that in Massera and Schäffer and corresponds to what they call a double exponential dichotomy induced by a disjoint dihedron (see [4, pp. 285–286]). If the matrix A is bounded, admissibility as defined here is equivalent to our definition of an exponential dichotomy. More refined results can be found in [4].)

Suppose rank $P_1 = m$, rank $P_2 = q = n - m$ where n is the dimension of the underlying space, and consider the pair of projections $(H_1(r), H_2(r))$ where $H_i(r) = Y(r)P_iY^{-1}(r)$. Note that the subspaces determined by these projections are carried into themselves by the solution operator $Y(t)Y^{-1}(s)$ to (1.2) (i.e. if $x \in$ Range $H_i(s)$, then

$$y = Y(t)Y^{-1}(s)x = Y(t)P_iY^{-1}(t)Y(t)Y^{-1}(s)x \in \text{Range } H_i(t)).$$

It is clear from (1.3) that the columns of $H_1(r)$ span the subspace of initial data at $t = r$ such that the solution to (1.2) is bounded as $t \longrightarrow +\infty$ (the stable manifold), while range$(H_2(r))$ is the space of initial data at $t = r$ such that the solution to (1.2) is bounded as $t \longrightarrow -\infty$. This is the generalization to nonautonomous systems of the case that A is a constant matrix whose eigenvalues have both positive and negative real parts.

As already stated, it is known that if A is bounded, an exponential dichotomy is equivalent to (L_∞, L_∞) admissibility. The boundedness of solutions to (1.1) can be regarded as boundary conditions at $t = \pm\infty$, and it seems reasonable to treat the numerical computation of the solution y_∞ as a singular two point boundary value problem. An algorithm to solve for the solution y_∞ is presented here. Below we give a heuristic description of the algorithm. In Section 2 convergence proofs are given for the case that (1.2) has an exponential dichotomy, and in Section 3 some numerical results are presented.

We point out that it is not necessary for the homogeneous equation to have an exponential dichotomy. One can apply the algorithm in any case where all solutions to the homogeneous equation must grow as t approaches the upper and lower endpoints of the interval. For example $A(t)$ would have a simple pole at a finite time t_0 and have some exponentially growing solutions as $t \longrightarrow \infty$. Under appropriate conditions on the behavior of the residue of A at t_0, all solutions to the homogeneous equation will be unbounded as t approaches one of the endpoints. Nevertheless, bounded solutions would exist for the inhomogeneous equation. Numerical results for such a system are given in Section 3. One can also hope to apply this procedure to homogeneous equations of the kind discussed by Keller [3, pp. 53–58] where one has inhomogeneous conditions at the lower endpoint and boundedness conditions at ∞.

Before describing the algorithm we give some definitions. By an m-dimensional

hyperplane in R_n we mean the set of vectors of the form $x = u + E\lambda$. Here u is an n-vector, E is an $m \times n$ matrix of rank m while λ is an arbitrary m-vector. The pair (u, E) need not be unique. In fact, we can take the columns of E to be orthonormal and u orthogonal to E. In this case it is easy to see that the columns of E are unique up to a rotation and the vector u is the projection of the hyperplane in the orthogonal complement of the space spanned by the columns of E and is therefore unique.

Any $m + 1$ vectors x_i, with rank at least m lie in a unique hyperplane. In fact, if the vectors x_i have rank $m + 1$, we can take $e_i = (x_i - x_{m+1})$ and u to be x_{m+1}. If the x_i have rank m and we assume $x_i, \ldots, x_m$ are linearly independent, we can take $e_i = x_i$ and $u = 0$, i.e. the hyperplane is a subspace.

An m-dimensional hyperplane described by $x = u + E\lambda$ and a q-dimensional hyperplane $z = w + D\gamma$ will intersect in exactly one point provided the vectors $\{e_i, d_j\}$ have rank n. The point of intersection can be gotten by solving the linear system

$$[E, -D]\begin{pmatrix}\lambda\\ \gamma\end{pmatrix} = w - u = \xi. \tag{1.4}$$

The columns of E and D determine a decomposition of R^n. Let L_1 and L_2 be the projections associated with this decomposition. Let $\|L_1\|, \|L_2\| \leqslant K_1$ in, for example, the Euclidean norm. It is easy to see that for the solution to (1.4) we have

$$\|\xi\| \geqslant \frac{1}{K_1}\max[\|E\lambda\|, \|D\gamma\|]. \tag{1.5}$$

The factor $1/K_1$ is a measure of the angular separation of these subspaces (see Massera and Schäffer, Chapter 1 for a more extended discussion of this concept) and is independent of the particular bases (e_i, d_j) used to represent the decomposition. If $\|E\lambda\| \geqslant \delta\|\lambda\|$, $\|D\gamma\| \geqslant \delta\|\gamma\|$, then from (1.5) it follows that

$$\max(\|\lambda\|, \|\gamma\|) \leqslant \frac{K_1}{\delta}\|\xi\|. \tag{1.6}$$

In theory one can always take $\delta = 1$ by separately orthogonalizing the columns of E and of D. But if these vectors in E and D are chosen "nearly dependent" so that δ is small, then (1.6) indicates that the system (1.4) can become ill-conditioned for numerical solution. This will be an important consideration in what follows.

We now describe the algorithm. Let I be a fixed compact interval $[a, b]$ on which one wants to approximate the unique bounded solution y_∞. Choose $t_r >> b$, $t_l << a$ and at $s = t_r$ prescribe $m + 1$ arbitrary initial vectors for the integration backwards of Eq. (1.1). These known vectors can now be represented in the form

$$\begin{aligned} &y_\infty(t_r) + v_{m+1} + e_i, \qquad i = 1, \ldots, m,\\ &y_\infty(t_r) + v_{m+1}, \end{aligned} \tag{1.7}$$

where we do not know the value of $y_\infty(t_r)$. Similarly, at $s = t_l$ prescribe $q + 1$ arbitrary initial vectors which we may write as

$$\begin{aligned} &y_\infty(t_l) + w_{q+1} + d_j, \qquad j = 1, \ldots, q,\\ &y_\infty(t_l) + w_{q+1}, \end{aligned} \tag{1.8}$$

where again the value of $y_\infty(t_l)$ is not known. The initial vectors in (1.7) are inte-

grated backwards to $t = a$ while the initial vectors in (1.8) are integrated forwards to $t = b$. At any instant t we consider the hyperplanes formed by these solutions. Defining $G_1(t, s) = Y(t)P_1 Y^{-1}(s)$, $G_2(t, s) = Y(t)P_2 Y^{-1}(s)$ these hyperplanes can be represented as

$$(1.9)\quad \begin{aligned} &\text{(a)}\quad S(t) = y_\infty(t) + G_1(t, t_r)v_{m+1} + G_1(t, t_r)E\lambda \\ &\qquad\qquad + G_2(t, t_r)v_{m+1} + G_2(t, t_r)E\lambda, \\ &\text{(b)}\quad U(t) = y_\infty(t) + G_2(t, t_l)w_{q+1} + G_2(t, t_l)D\gamma \\ &\qquad\qquad + G_1(t, t_l)w_{q+1} + G_1(t, t_l)D\gamma, \end{aligned}$$

for arbitrary $\lambda \in R_m$, $\gamma \in R_q$. Now if T is a sufficiently large number with $T \leqslant \min[\operatorname{dist}(t_r, I), \operatorname{dist}(t_l, I)]$, then $G_2(t, t_r)$, $G_1(t, t_l) = O(e^{-\alpha T})$ and unless the initial data are badly chosen (for example $G_1(t, t_r)E = 0$) one would expect these hyperplanes to be close to the hyperplanes

$$(1.10)\quad \begin{aligned} S_1(t) &= y_\infty(t) + G_1(t, t_r)v_{m+1} + G_1(t, t_r)E\lambda, \\ U_1(t) &= y_\infty(t) + G_2(t, t_l)w_{q+1} + G_2(t, t_l)D\gamma, \end{aligned}$$

which describe the unstable (respectively, in the backwards and forward direction of t) manifolds of the solutions of the differential equation. Thus, the intersection $S(t) \cap U(t)$ will be close to the unique point of the intersection $S_1(t) \cap U_1(t)$. It will be shown in Section 2 that except for initial data chosen from a set of zero measure there must exist λ_0 and γ_0 such that

$$G_1(t, t_r)v_{m+1} + G_1(t, t_r)E\lambda_0 = 0, \qquad G_2(t, t_l)w_{q+1} + G_2(t, t)D\gamma_0 = 0;$$

and thus, this intersection must be $y_\infty(t)$.

In practice, one cannot carry out these integrations because the solutions, for example $G_1(t, t_r)E$, will grow exponentially as $t \longrightarrow -\infty$. One must, therefore, compute the hyperplane $S(t)$ at times t_j and take $m + 1$ new values in $S(t_j)$ of smaller norm to continue the integration. As long as the new vectors have rank at least m, one does not change $S(t)$ and the intersection of the two hyperplanes will be the same. However, if one should choose new vectors which are nearly dependent, one could obtain a poorly conditioned linear system as described above. The safest way to do this is to orthonormalize the vectors $Y(t)Y^{-1}(t_r)e_i$ and $Y(t)Y^{-1}(t_l)d_j$ (i.e. one gets an orthogonal basis for the hyperplanes $S(t)$ and $U(t)$) which is the method of Godunov and Conte, etc. (see Keller [3, p. 7]); other techniques are possible as described in Section 3, but one must bear in mind that with any of these methods the vectors may become numerically dependent.

We point out that the crucial part of the algorithm is the hyperplanes $S(b)$ and $U(a)$. From these hyperplanes one can compute regular two-point boundary conditions and use any scheme to solve two-point boundary value problems. Finding the hyperplanes $S(t)$, $U(t)$ and the intersection $S(t) \cap U(t)$ for $t \in [a, b]$ corresponds to a double shooting scheme. Other, more efficient techniques are available (see [3, pp. 2–8]) although the double shooting scheme is the simplest scheme to implement as the same

code for the backwards integration is used for $t \in [a, b]$ and similarly for the forward integration.

II. Convergence Proof. Here we give a theoretical justification of the algorithm described in the preceding section. For simplicity the proofs are given for the continuous case. It will be seen that the same proof is valid in the case of discretization by a strongly stable difference scheme.

Let $I = [a, b]$ and let $t_r > b$, $t_l < a$ be chosen so that $T = \min(\operatorname{dist}(t_r, I), \operatorname{dist}(t_l, I))$ is sufficiently large. Let the initial vectors in (1.7) and (1.8) be chosen at random in a certain ball. Since $y_\infty(t)$ is bounded, it is sufficient to suppose that the vectors v_{n+1}, e_j, w_{q+1}, d_i lie in some ball. We use the notation B_n^R for the ball of radius R in R_n. Thus, we have

$$v_{n+1}, e_j, w_{q+1}, d_i \in B_n^R, \qquad j = 1, \ldots, m, i = 1, \ldots, q,$$

for some R. (All norms will be taken as Euclidean unless otherwise stated.)

Now let E be a vector in R_{mn} and represent it by the $m \times n$ matrix with columns e_j. Similarly, let $D \in R_{qn}$ be represented by the columns d_i. We thus have

$$E \in B_{mn}^{\sqrt{n}R}, \qquad D \in B_{qn}^{\sqrt{q}\bar{R}}.$$

We will prove the following theorem.

THEOREM 1. *There exist functions $T(\epsilon)$ and $\mu(\epsilon)$ with $T(\epsilon) \longrightarrow \infty$ $(\epsilon \longrightarrow 0)$, $\mu(\epsilon) \longrightarrow 0$ $(\epsilon \longrightarrow 0)$ so that for any interval I the difference $\|y_\infty(t) - S(t) \cap U(t)\| < \epsilon$ for $t \in I$, if $T \geqslant T(\epsilon)$ unless E and D are chosen from a set of measure $< \mu(\epsilon)$.*

We note that $T(\epsilon)$ and $\mu(\epsilon)$ are independent of the interval I and of the starting points t_r and t_l. The only part of the proof that is not straightforward is the proof that the measure of the "bad" set can be bounded independently of the starting points t_r and t_l.

We first prove the following lemma.

LEMMA 1. *Let $E \in R_{mn}$ be represented by an $m \times n$ matrix and let $E \in B_{mn}^{R_1}$ for some R_1. Let P_r be a family of m-dimensional projections which are uniformly bounded in r, i.e., $\|P_r\| \leqslant C$. For any δ let $S(\delta, P_r) = \{E \in B_{mn}^{R_1}: \|P_r E\lambda\| \geqslant \delta\|\lambda\|$ for all $\lambda \in R_m\}$ and let $C(\delta, P_r)$ be its complement. If μ denotes the measure in R_{mn} generated by the Euclidean metric, then*

$$\mu(C(\delta, P_r)) \longrightarrow 0 \qquad (\delta \longrightarrow 0) \text{ uniformly in } r. \tag{2.1}$$

For our purposes the projections $P_i(r)$ will be the $H_i(r)$ defined in Section 1. If r is fixed, then $C(\delta, P_r)$ is the set of E such that for some λ with $\|\lambda\| = 1$ we have $\|P_r E\lambda\| \leqslant \delta$. Thus, as $\delta \longrightarrow 0$, $u(C(\delta, P_r)) \longrightarrow u(C^1)$ where C^1 is the set of $m \times n$ matrices such that for some λ we have $P_r E\lambda = 0$. Since r is fixed, we can always assume

$$P_r = \begin{pmatrix} I_{m \times m} & 0 \\ 0 & 0 \end{pmatrix}$$

and, thus, $u(C^1) = 0$. Thus, the crucial point of Lemma 1 is the uniformity in r.

PROOF. We can obviously take $R_1 = 1$. It is clear that $\mu(C(\delta, P_r)) \longrightarrow 0$ $(\delta \longrightarrow 0)$ for each fixed r, and it is the uniformity which must be proved. Let $S_r = I - P_r$, and write E as $E_1 + E_2$ with $E_1 = P_r E$, $E_2 = S_r E$. This is a direct sum decomposition, and we can define a new norm (non-Euclidean) by

$$\|E\|_r = \max(\|P_r E\|, \|S_r E\|).$$

From the uniform boundedness of the P_r we can find a Q independent of r such that

$$\|E\|_r \leqslant Q\|E\|, \qquad \|E\| \leqslant Q\|E\|_r.$$

The new norm $\| \ \|_r$ induces a new measure μ_r on R_{mn} and $\mu(S) \leqslant Q\mu_r(S)$ for any Borel set S. It is thus sufficient to show that $\mu_r C((\delta, P_r))$ converges uniformly to zero for $\|E\|_r \leqslant Q$.

Let $p_1, \ldots, p_m$ be any orthonormal basis spanning the range of P_r while $s_1, \ldots, s_q$ will be an orthonormal basis spanning the range of S_r. If E is decomposed as $E_1 + E_2$ with $E_1 = P_r E$, $E_2 = S_r E$, we can write

$$E_1 = PA, \qquad E_2 = SB,$$

where P is the matrix of columns $p_1, \ldots, p_m$ while S is the matrix of columns $s_1, \ldots, s_q$, and A and B are $m \times m$ and $q \times m$ matrices, respectively. This clearly defines a mapping of $R_{mn} \longrightarrow R_{mm} \otimes R_{qm}$ which becomes an isometry (hence measure preserving), if we give it the norm

$$\|(A, B)\|_2 = \max(\|A\|, \|B\|).$$

Now $E \in C(\delta, P_r)$ if and only if there exists $\lambda \in R_m$ with $\|P_r E\lambda\| < \delta\|\lambda\|$. This is equivalent to $\|A\lambda\| \leqslant \delta\|\lambda\|$. Now the measure of the set of such points in the ball of radius Q in $R_{mm} \times R_{qm}$ is $o(1)$ as $\delta \longrightarrow 0$ and is independent of P_r. This completes the proof of the lemma.

We can now complete the proof of the theorem. Recalling the definition of $G_1(t, s)$ and $G_2(t, s)$ we have from the definition of an exponential dichotomy the decay estimates

$$\begin{aligned} \|G_1(t, s)\| &\leqslant Ke^{-\alpha(t-s)}, \quad t \geqslant s, \\ \|G_2(t, s)\| &\leqslant Ke^{-\alpha(s-t)}, \quad s \geqslant t. \end{aligned} \tag{2.2}$$

We now obtain estimates from below. In fact, if $s \geqslant t$ and we set $y = G_1(t, s)x$, we obtain, with H_1 defined as in Section 1, $H_1(s)x = G_1(s, t)y$; and we can thus write

$$\|G_1(t, s)x\| \geqslant \frac{1}{K} e^{\alpha(s-t)}\|H_1(s)x\|, \qquad s \geqslant t; \tag{2.3}$$

and similarly for G_2

$$\|G_2(t, s)x\| \geqslant \frac{1}{K} e^{\alpha(t-s)}\|H_1(s)x\|, \qquad t \geqslant s. \tag{2.4}$$

Now consider the representation for the hyperplanes $S(t)$ and $U(t)$ given in (1.9). We first eliminate the troublesome terms $G_1(t, t_r)v_{m+1}$ and $G_2(t, t_l)w_{q+1}$ so that the form of $S(t)$ and $U(t)$ are approximately $y_\infty(t) + E\lambda$ and $y_\infty(t) + D\gamma$, respectively.

If E is chosen so that $\|H_1(t_r)E\lambda\| \geqslant \delta\|\lambda\|$ where $\delta > 0$ is to be specified, we can solve the equation $H_1(t_r)v_{m+1} = H_1(t_r)E\lambda_1$ for λ_1. Similarly, choosing D in this way, we can solve the equation $H_2(t_l)w_{q+1} = H_2(t_l)D\gamma_1$ for γ_1. If we use C as a generic constant depending only on R and K, we will have

$$\max(\|\lambda_1\|, \|\gamma_1\|) \leqslant C/\delta.$$

Now define $\lambda' = \lambda + \lambda_1$, $\gamma' = \gamma + \gamma_1$ so that the hyperplanes $S(t)$ and $G(t)$ can be represented as

(a) $S(t) = y_\infty(t) + G_1(t, t_r)E\lambda' + G_2(t, t_r)E\lambda' + G_2(t, t_r)[v_{m+1} - E\lambda_1]$,

(b) $U(t) = y_\infty(t) + G_2(t, t_l)D\gamma' + G_1(t, t_l)D\gamma' + G_1(t, t_l)[w_{q+1} - D\gamma_1]$.

At the point of the intersection λ' and γ' must satisfy the following linear equation

(a) $$G_1(t, t_r)E\lambda' = G_1(t, t_l)D\gamma' + G_1(t, t_l)[w_{q+1} - D\gamma_1],$$

(b) $$G_2(t, t_l)D\gamma' = G_2(t, t_r)E\lambda' + G_2(t, t_r)[v_{m+1} - E\lambda_1].$$

From (2.2), (2.3) and (2.4) we obtain the following estimates:

$$e^{\alpha T}\|\delta'\| \leqslant Ce^{-\alpha T}[\|\gamma'\| + 1/\delta], \qquad e^{\alpha T}\|\delta'\| \leqslant Ce^{-\alpha T}[\|\lambda'\| + 1/\delta].$$

So far δ has remained unspecified. We now set $\delta = e^{-\alpha T/2}$ and obtain

$$\max(\|\lambda'\|, \|\gamma'\|) \leqslant Ce^{-\alpha T}.$$

From this estimate we obtain at the point of intersection

$$\|S(t) - y_\infty(t)\| \leqslant Ce^{-\alpha T/2}. \tag{2.5}$$

Theorem 1 now follows from choosing T so that the right-hand side of (2.5) is $< \epsilon$, setting $\delta = e^{-\alpha T/2}$ and using Lemma 1 to bound the set of initial values where (2.5) fails.

The preceding was given for the continuous case; however, one can see that it is a purely formal proof depending only on the existence of an exponential dichotomy. This permits one to use results of Bayliss [1] to extend Theorem 1 to strongly stable difference approximations.

In fact, let (1.1) be approximated by the l-step scheme

$$\sum_{j=0}^{l} \alpha_j y_{n+j} = k \sum_{j=0}^{l} \beta_j \dot{y}_{n+j}, \tag{2.6}$$

where $k = \Delta t$ and $\dot{y}$ is given by (1.1).

Associated with (2.6) we have the polynomials

$$\rho(x) = \sum_{j=0}^{l} \alpha_j x^j, \qquad \sigma(x) = \sum_{j=0}^{l} \beta_j x^j.$$

From consistency we know that $x_1 = 1$ is a simple root of $\rho(x) = 0$. Strong stability is a restriction on the other roots x_u, $u = 2, \ldots, l$. Specifically, we have (see, for instance, Gear [2])

$$|x_u| < 1, \qquad u = 2, \ldots, l.$$

Since we want to solve (2.6) backwards as well as forwards, we also require $x_u \neq 0$.

We convert (2.6) into a 1-step scheme in the usual way by defining the vector $w_n = (y_{n+l} - 1, \ldots, y_n)^T$. Using (1.1) to substitute for $\dot{y}$, we obtain the linear inhomogeneous 1-step difference equation

$$w_{n+1} = U_n w_n + k\tilde{f}_n. \tag{2.7}$$

Here U_n (the companion matrix) is an $ln \times ln$ matrix while w_n and $\tilde{f}_n$ are ln-vectors. Under these conditions it has been shown (see Bayliss [1]), that for k sufficiently small:

(i) (2.7) has a unique bounded solution w_n for any bounded forcing term $\tilde{f}_n$. If $\tilde{f}_n$ comes from discretizing a function f as in (1.1) and $w_{\infty,n}$ is defined as $(y_\infty((n+l-1)k), \ldots, y_\infty(nk))^T$, then $w_n - w_{\infty,n} \longrightarrow 0$ $(k \longrightarrow 0)$ uniformly in n; and in fact, $\|w_n - w_{\infty,n}\|_\infty = O(k^p)$ where p is the order of the scheme (2.6) under sufficient smoothness conditions on A and f.

(ii) Let W_n be the fundamental matrix solution to the homogeneous version of (2.7) $(W_0 = I)$. Then there exist projections $P_1'(k)$, $P_2'(k) = I - P_1'(k)$ of dimension $n(l-1) + m$, q respectively, and constants K_1 and α_1 independent of k such that

$$\|W_n P_1' W_j^{-1}\| \leqslant K_1 e^{-\alpha k(n-j)}, \qquad n \geqslant j,$$

$$\|W_n P_2' W_j^{-1}\| \leqslant K_1 e^{-\alpha k(j-n)}, \qquad j \geqslant n.$$

This, of course, is exactly the discrete analogue to an exponential dichotomy. Note that the dimension of the stable manifold is increased by $n(l-1)$ to take into account the roots x_u inside the unit circle.

Under these conditions, it is clear that the proof given in Theorem 1 will be valid for the difference equation (2.7) to approximate the solution $w_{\infty,n}$ which in turn is an approximation to the solution y to (1.1).

III. Numerical Results. Here we describe some numerical tests of the scheme of Section I. Tests were run on a three-dimensional system

$$\dot{y} = Ay + f. \tag{3.1}$$

The matrix A was chosen so that the fundamental solution to the homogeneous equation was

$$Y(t) = U_2(t)U_1(t)e^{Et}$$

with

$$U_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin w_2 t & -\cos w_2 t \\ 0 & \cos w_2 t & \sin w_2 t \end{pmatrix}, \qquad U_1(t) = \begin{pmatrix} \cos w_1 t & \sin w_1 t & 0 \\ \sin w_1 t & -\cos w_1 t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}. \tag{3.2}$$

The constants w_1 and w_2 were chosen as 1 and $\sqrt{3}$, respectively. One can verify that E has eigenvalues $-1, -2, +1$ so that the homogeneous version of (3.1) has an exponential dichotomy with a stable manifold of rank 2 and an unstable manifold of rank 1. The forcing term is chosen so that $y_\infty(t) = (\sin t, \cos\sqrt{2}t, 0)^T$ is the unique bounded solution to (3.1). The first order Euler scheme $y_{n+1} = y_n + \Delta t \dot{y}_n$ and the second order Crank-Nicholson scheme $y_{n+1} = y_n + \Delta t[y_n + y_{n+1}]/2$ were used for the integrations. Renormalization was done by writing the vector y as $(y_1, y_2, y_3)^T$ and solving for the hyperplane as $y_3 = Ay_1 + By_2 + C$. New vectors were then chosen to minimize the three functionals $y_1^2 + y_2^2 + y_2^2$, $y_1^2 + .1y_2^2 + y_3^2$ and $(y_1 + 1)^2 + y_2^2 + y_3^2$. This procedure is not guaranteed to produce at least two independent vectors, although one might expect it to work except in pathological situations. Indeed, in practice the method works as we now indicate. Initial data were generated by a random number routine normalized to lie in the interval $[-5, 5]$.

Table I gives relative L_2 errors for y over the interval $[0, 1]$, for different time steps, for the Euler scheme, while Table II gives the errors for the Crank-Nicholson scheme. We note the linear and quadratic, respectively, rates of convergence. Experiments with a fixed time step and different sets of initial data produced results agreeing to three significant figures.

Δt	L_2 error
.04	.027
.02	.013
.0133	.0087
.01	.0065

TABLE I. *Errors for Eq.* (3.1) *with Euler schemes*

Δt	L_2 error
.04	.00020
.02	.000049
.0133	.000022
.01	.000012

TABLE II. *Errors for Eq.* (3.1) *with Crank-Nicholson scheme*

Integrations were also done for the homogeneous equation

$$\dot{y} = Ay. \tag{3.3}$$

Here two conditions were specified at $t = 0$ and we solved for the unique solution bounded as $t \longrightarrow \infty$. This type of system is discussed by Keller [3], as mentioned previously, but his method is valid only when A is asymptotically constant. Here the sought for solution was

$$U_2(t)U_1(t)e^{Et}\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

where U_2, U_1 and E are given in (3.2). The values of y_1 and y_2 at $t = 0$ were prescribed and y_3 together with data at a large time ($t = 18$) were chosen randomly. L_2 errors over the interval [0, 1] are given in Tables III and IV for the Euler and Crank-Nicholson schemes, respectively. We again note the expected linear and quadratic convergence.

Δt	L_2 error
.04	.043
.02	.021
.0133	.014
.01	.010

TABLE III. *Errors for Eq.* (3.3) *with Euler scheme*

Δt	L_2 error
.04	.00088
.02	.00022
.0133	.000098
.01	.000055

TABLE IV. L_2 *errors for Eq.* (3.3) *with Crank-Nicholson scheme*

Tests were also run on a two-dimensional system of the form

$$\frac{dy}{dr} = \frac{1}{r}Jy + pDy + f(r) \tag{3.4}$$

with

$$J = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad D = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$$

and $p > 0$.

Equation (3.4) results from applying the Laplace transform in time to a hyperbolic wave type equation of acoustics, the study of which is still in progress. The homogeneous equation has no nontrivial bounded solutions on $(0, \infty)$ and instead of boundedness one can require that the solution to (3.4) grow slower than $1/r$ as $r \longrightarrow 0$ and less than exponentially as $r \longrightarrow \infty$. The forcing term was chosen so that $y_\infty = (\sqrt{r}, 0)$ was the sought for solution. Table V indicates normalized L_2 errors over the interval [1, 2] for different values of p and different time steps. The Euler scheme was used for the integrations, and we note the expected rate of convergence. The case

of small p is most important since it corresponds to large time in the inverse Laplace transform. The asymptotic behavior for small p is not maintained as Δr is decreased because the integration to get the stable manifold must be started at a large r_0 and round-off error then dominates the discretization error.

P	Δr	L_2 error
.05	.05	.015
.05	.1	.0027
.05	0.2	.0063
.35	.05	.0022
.35	.1	.0020
.35	.2	.0055
1	.05	.00087
1	.1	.0021
10	.0062	.0000496
10	.0125	.0000833
20	.0062	.0000233
20	.0125	.0000421

TABLE V. L_2 *errors for Eq.* (3.4)

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