An Iterative Process for Nonlinear Monotone Nonexpansive Operators in Hilbert Space

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Abstract. The following theorem is proved: Suppose $H$ is a complex Hilbert space, and $T: H \to H$ is a monotone, nonexpansive operator on $H$, and $f \in H$. Define $S: H \to H$ by $Su = -Tu + f$ for all $u \in H$. Suppose $0 \leq t_n < 1$ for all $n = 1, 2, 3, \ldots$, and $\sum_{n=1}^\infty t_n(1-t_n)$ diverges. Then the iterative process $V_{n+1} = (1-t_n)V_n + t_nSV_n$ converges to the unique solution $u = p$ of the equation $u + Tu = f$.

It is well known that the equation $u + Tu = f$ has a unique solution $u$ for each $f$ in a Hilbert space $H$ provided that $T: H \to H$ is monotone and Lipschitzian (e.g., see [3]). The purpose of this paper is to show that if $T$ is nonexpansive (Lipschitz constant 1), then the Mann iterative process [1] will, under a certain condition, converge to this unique solution.

The normal Mann iterative process is defined by $V_{n+1} = (1-t_n)V_n + t_nTV_n$. We will use the condition that $\sum_{n=1}^\infty t_n(1-t_n)$ diverges, which has been extensively used by Groetsch [2].

Theorem. Suppose $H$ is a complex Hilbert space, and $T: H \to H$ is a monotone, nonexpansive operator on $H$, and $f \in H$. Define $S: H \to H$ by $Su = -Tu + f$ for all $u \in H$. Suppose $0 \leq t_n < 1$ for all $n = 1, 2, 3, \ldots$, and $\sum_{n=1}^\infty t_n(1-t_n)$ diverges. Then the iterative process $V_{n+1} = (1-t_n)V_n + t_nSV_n$ converges to the unique solution $u = p$ of the equation $u + Tu = f$.

Proof. We first observe that $S$ is nonexpansive and satisfies $\text{Re}(Sx - Sy, x - y) \leq 0$ for all $x, y \in H$. Since $Sp = p$, we get

$$\|V_{n+1} - p\|^2 = \|(1-t_n)(V_n - p) + t_n(SV_n - Sp)\|^2$$

$$= (1-t_n)^2\|V_n - p\|^2 + 2t_n(1-t_n)\text{Re}(SV_n - Sp, V_n - p)$$

$$+ t_n^2\|SV_n - Sp\|^2.$$ 

Using $\text{Re}(SV_n - Sp, V_n - p) \leq 0$, $t_n(1-t_n) > 0$, and $\|SV_n - Sp\| \leq \|V_n - p\|$, we get

$$\|V_{n+1} - p\|^2 \leq ((1-t_n)^2 + t_n^2)\|V_n - p\|^2,$$

which can also be written

$$\|V_{n+1} - p\|^2 \leq (1-2t_n(1-t_n))\|V_n - p\|^2.$$
Upon iteration this yields
\[
\|V_{n+1} - p\|^2 \leq \left\{ \prod_{k=1}^{n} \left[ 1 - 2t_k(1 - t_k) \right] \right\} \|V_1 - p\|^2.
\]

We note that \(0 < 2t(1 - t) < \frac{1}{2}\) for \(0 < t < 1\). From the divergence of \(\sum_{n=1}^{\infty} t_n(1 - t_n)\) it now follows that \(\lim_n \|V_{n+1} - p\| = 0\), whence \(\{V_n\}\) converges to \(p\).

A particular case is of some interest, viz. \(t_n = 1/n\). \((1/n)(1 - 1/n) = (n - 1)/n^2 > 1/2n\) for \(n > 2\) establishes the divergence of \(\sum_{n=1}^{\infty} t_n(1 - t_n)\). There is however an alternate method in this particular case which gives the additional information of an error estimate. As before, we let \(p\) denote the unique solution of \(u + Tu = f\), and we observe that
\[
\|SV_n - Sp\| < \|V_n - p\| < \|V_1 - p\|.
\]

We have
\[
V_{n+1} = \frac{n}{n+1} V_n + \frac{1}{n+1} SV_n
\]
and so
\[
V_{n+1} - p = \frac{n}{n+1} (V_n - p) + \frac{1}{n+1} (SV_n - Sp),
\]
whence
\[
\|V_{n+1} - p\|^2 = \frac{n^2}{(n+1)^2} \|V_n - p\|^2 + \frac{2n}{(n+1)^2} \text{Re}(SV_n - Sp, V_n - p)
\]
\[
+ \frac{1}{(n+1)^2} \|SV_n - Sp\|^2.
\]
Thus, we get
\[
(n+1)^2 \|V_{n+1} - p\|^2 - n^2 \|V_n - p\|^2 \leq \|V_1 - p\|^2.
\]
The left-hand side collapses upon summation from \(n = 1\) to \(n = N\) to yield
\[
(N+1)^2 \|V_{N+1} - p\|^2 - \|V_1 - p\|^2 < N \cdot \|V_1 - p\|^2.
\]
Hence for each \(N = 1, 2, 3, \ldots\), we have
\[
\|V_{N+1} - p\|^2 \leq (1/(N+1)) \|V_1 - p\|^2.
\]
Thus \(\{V_n\}\) converges to \(p\) and for each \(n\) we have
\[
\|V_{n+1} - p\| \leq \frac{1}{\sqrt{n+1}} \|V_1 - p\|.
\]