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Odd Integers N With Five Distinct Prime Factors for Which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$

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Abstract. We make a table of odd integers N with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$, and show that for such N $|\sigma(N)/N - 2| > 10^{-14}$. Using this inequality, we prove that there are no odd perfect numbers, no quasiperfect numbers and no odd almost perfect numbers with five distinct prime factors. We also make a table of odd primitive abundant numbers N with five distinct prime factors for which $2 < \sigma(N)/N < 2 + 2/10^{10}$.

1. A positive integer N is called perfect, quasiperfect (QP), or almost perfect according as $\sigma(N) = 2N$, $2N + 1$, or $2N - 1$, respectively, where $\sigma(N)$ is the sum of the positive divisors of N. While twenty-four even perfect numbers are known, no odd perfect (OP) numbers, no QP numbers, and no almost perfect numbers except a power of 2 are known.

In this paper we make a table of odd integers N with five distinct prime factors for which

$$2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}, \tag{1}$$

and we show that for such N

$$|\sigma(N)/N - 2| > 10^{-14}.$$

Using this inequality, we prove that there are no OP, QP, or odd almost perfect (OAP) numbers with five distinct prime factors.

N is called primitive abundant if N is abundant ($\sigma(N) > 2N$) and every proper divisor M of N is deficient ($\sigma(M) < 2M$). In 1913 Dickson [4] published a table of odd primitive abundant numbers with less than five distinct prime factors. In this paper we also make a table of odd primitive abundant numbers N with five distinct prime factors for which

$$2 < \sigma(N)/N < 2 + 2/10^{10}. \tag{2}$$

2. Throughout this paper we let $N = \prod_{i=1}^{r} p_i^{a_i}$ where $3 \leqslant p_1 < \cdots < p_r$ are primes and a_i's are positive integers. $p_i^{a_i}$ is called a component of N.

We define

$$a(p) = \min\{a \mid p^{a+1} > 10^{12}\},$$
$$\omega(N) = r,$$
$$S(N) = \sigma(N)/N = \prod_{i=1}^{r} (p_i^{a_i+1} - 1)/p_i^{a_i}(p_i - 1),$$

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$$A(N) = \left[\prod_{a_i < a(p_i)} S(p_i^{a_i})\right]\left[\prod_{a_i \geqslant a(p_i)} S(p_i^{a(p_i)})\right],$$

$$B(N) = \left[\prod_{a_i < a(p_i)} S(p_i^{a_i})\right]\left[\prod_{a_i \geqslant a(p_i)} p_i/(p_i - 1)\right],$$

$$L(p^a) = \begin{cases} [10^{12} \log S(p^a)]/10^{12} & \text{if } a < a(p), \\ [10^{12} \log p/(p-1)]/10^{12} & \text{if } a \geqslant a(p), \end{cases}$$

where [ ] is the greatest integer function. We note that if p, q are primes with $p > q$ and a, b are positive integers then

$$S(p^a) = (p^{a+1} - 1)/p^a(p - 1) < p/(p - 1) = \lim_{a\to\infty} S(p^a) \leqslant (q + 1)/q \leqslant S(q^b),$$

and so $L(p^a) \leqslant L(q^b)$ and $A(N) \leqslant S(N) \leqslant B(N)$. Hence, we have

LEMMA 1. (a) *If* $A(N) > 2 - 10^{-12}$ *and* $B(N) < 2 + 10^{-12}$, *N satisfies* (1).

(b) *If* $A(N) \leqslant 2 - 10^{-12} < B(N) < 2 + 10^{-12}$, *some N satisfies* (1).

(c) *If* $2 - 10^{-12} < A(N) < 2 + 10^{-12} \leqslant B(N)$, *some N satisfies* (1).

(d) *If* $A(N) < 2 - 10^{-12}$ *and* $2 + 10^{-12} < B(N)$, *some N may satisfy* (1).

(e) *If* $2 + 10^{-12} < A(N)$ *or* $B(N) < 2 - 10^{-12}$, *N does not satisfy* (1).

In Lemmas 2 through 5 we assume that N satisfies (1) and $\omega(N) = 5$.

LEMMA 2.

$$0.6931471805544 < \sum_{i=1}^{5} L(p_i^{b_i}) < 0.6931471805655, \tag{3}$$

where $b_i = \min\{a_i, a(p_i)\}$.

Proof. Suppose p^a is a component of N. If $a < a(p)$, then

$$|\log S(p^a) - L(p^a)| < 10^{-12}.$$

If $a \geqslant a(p)$, then $p^{a+1} > 10^{12}$ and

$$10^{-12} > \log p/(p - 1) - L(p^a) > \log S(p^a) - L(p^a) \geqslant \log S(p^a) - \log p/(p - 1)$$
$$= \log(1 - 1/p^{a+1}) = -\sum_{i=1}^{\infty} 1/i(p^{a+1})^i > -1/(p^{a+1} - 1) \geqslant -10^{-12}.$$

Hence

$$|\log S(p^a) - L(p^a)| < 10^{-12}.$$

Since (1) holds,

$$0.6931471805544 < \log(2 - 10^{-12}) - 5/10^{12}$$
$$< \sum_{i=1}^{5} \log S(p_i^{a_i}) - 5/10^{12} < \sum_{i=1}^{5} L(p_i^{b_i})$$
$$< \sum_{i=1}^{5} \log S(p_i^{a_i}) + 5/10^{12} < \log(2 + 10^{-12}) + 5/10^{12}$$
$$< 0.6931471805655. \quad \text{Q.E.D.}$$

LEMMA 3. $p_1 = 3, p_2 \leqslant 11$ *and* $p_3 \leqslant 41$.

Proof. Lemma 3 follows from the following inequalities:

$$\frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16} < 2 - 10^{-12},$$

$$\frac{3}{2}\frac{13}{12}\frac{17}{16}\frac{19}{18}\frac{23}{22} < 2 - 10^{-12},$$

$$\frac{3}{2}\frac{5}{4}\frac{43}{42}\frac{47}{46}\frac{53}{52} < 2 - 10^{-12}. \quad \text{Q.E.D.}$$

LEMMA 4. $p_4 < 5000$.

Proof. Suppose N satisfies (1) and $p_4 \geqslant 5003$. Then

$$\begin{aligned} 0 &\leqslant L(p_5^{b_5}) \leqslant L(p_4^{b_4}) < \log S(p_4^{b_4}) + 10^{-12} \\ &< \log p_4/(p_4 - 1) + 10^{-12} < 1/(p_4 - 1) + 10^{-12} \\ &< 0.0002. \end{aligned}$$

Hence by (3)

$$0.69274 < \sum_{i=1}^{3} L(p_i^{b_i}) < 0.69315. \tag{4}$$

A computer (PDP11 at the University of Toledo) was used to find $\Pi_{i=1}^3 p_i^{b_i}$ satisfying (4), but there were none. Q.E.D.

Similarly, we can prove

LEMMA 5. $p_5 < 3000000$, *or* $\Pi_{i=1}^4 p_i^{b_i} = 3^7 5^6 17^2 233$ *and* $36549767 \leqslant p_5 \leqslant 36551083$.

The computer was used to find $N = \Pi_{i=1}^5 p_i^{a_i}$ satisfying $a_i \leqslant a(p_i)$, Lemmas 3, 4, 5, and Lemma 2 or Lemma 1(b), (c), (d), with the result given in Table 1.

LEMMA 6. *Suppose* $N = \Pi_{i=1}^5 p_i^{a_i}$ *and* $M = \Pi_{i=1}^5 p_i^{b_i}$ *where* $b_i = \min\{a_i, a(p_i)\}$. *If* $M = 3^{23} 5^{12} 17^6 257^4 65521$, $|S(N) - 2| > 5/10^{13}$;
if $M = 3^8 5^{14} 17^3 251 \cdot 1884529$, $|S(N) - 2| > 2/10^{14}$;
if $M = 3^8 5^9 17^3 251 \cdot 1579769$, $|S(N) - 2| > 3/10^{13}$;
if $M = 3^8 5^8 17^9 269^4 4153^3$, $|S(N) - 2| > 4/10^{14}$;
if $\Pi_{i=1}^4 p_i^{b_i} = 3^7 5^6 17^2 233$, $|S(N) - 2| > 10^{-14}$.
In all other cases $|S(N) - 2| > 10^{-13}$.

Proof. The first part of Lemma 6 follows from the following inequalities:

$$S(3^{23} 5^{12} 17^6 257^4 65521) < 2 - 5/10^{13},$$
$$S(3^{23} 5^{12} 17^6 257^5 65521) > 2 + 1/10^{12},$$
$$S(3^8 5^{14} 17^3 251)\ 1884529/1884528 < 2 - 2/10^{14},$$
$$S(3^8 5^9 17^3 251 \cdot 1579769) < 2 - 4/10^{13},$$
$$S(3^8 5^9 17^3 251 \cdot 1579769^2) > 2 + 3/10^{13},$$
$$S(3^8 5^8 269^4)\ 17/16 \cdot 4153/4152 < 2 - 4/10^{14},$$
$$S(3^8 5^8 17^9 269^5 4153^3) > 2 + 3/10^{13},$$
$$S(3^7 5^6 17^2 233 \cdot 36550379) > 2 + 5/10^{14},$$

and

$$S(3^7 5^6 17^2 233)\ 36550429/36550428 < 2 - 10^{-14}.$$

Suppose $|S(N) - 2| \leqslant 10^{-13}$. Then (1) holds, and so N is given in Table 1; however, for every N in Table 1 except for those given above $S(N) \leqslant B(N) < 2 - 10^{-13}$, or $S(N) \geqslant A(N) > 2 + 10^{-13}$. Q.E.D.

We have proved

THEOREM. *If N is an odd integer with $\omega(N) = 5$, $|\sigma(N)/N - 2| > 10^{-14}$.*

3. We used a similar method to find odd primitive abundant numbers $N = \Pi_{i=1}^5 p_i^{a_i}$ for which (2) holds, with the result given in Table 2 in the microfiche. Table 2 includes odd primitive abundant numbers N with $\omega(N) = 5$ one of whose component p^a is greater than 10^{10}; for, letting $M = N/p^a$, we have

$$\begin{aligned} 2 < \sigma(N)/N &= \sigma(M)\sigma(p^a)/Mp^a = \sigma(M)(p\sigma(p^{a-1}) + 1)/Mp^a \\ &= \sigma(Mp^{a-1})/Mp^{a-1} + \sigma(M)/Mp^a < 2 + 2/10^{10}, \end{aligned}$$

showing that (2) holds.

4. Suppose N is an odd integer such that $\sigma(N) = 2N + A$. If $|A/N| \leqslant 10^{-14}$, then by our Theorem $\omega(N) \geqslant 6$. We give three examples of such N.

Suppose N is OP. Sylvester (1888), Dickson (1913), and Kanold (1949) proved that $\omega(N) \geqslant 5$. From our Theorem we have

PROPOSITION 1. *If N is OP, $\omega(N) \geqslant 6$.*

This fact was also proved by Gradštein (1925), Kühnel (1949) and Webber (1951). Pomerance [1] (1972) and Robbins (1972) proved that $\omega(N) \geqslant 7$, and Hagis [2] proved that $\omega(N) \geqslant 8$.

PROPOSITION 2. *If N is QP, $\omega(N) \geqslant 6$.*

Proof. By [3] if N is QP, then N is an odd perfect square, $\omega(N) \geqslant 5$ and $N > 10^{20}$. Hence $2 < S(N) = 2 + 1/N < 2 + 10^{-20}$, and so by Theorem $\omega(N) \geqslant 6$. Q.E.D.

LEMMA 7. *If N is OAP, pN is primitive abundant for some $p|N$.*

Proof. Suppose $N = \Pi_{i=1}^r p_i^{a_i}$ is OAP, and choose j so that $\sigma(p_j^{a_j}) \geqslant \sigma(p_i^{a_i})$ for every i. Letting $p = p_j$, $a = a_j$ and $L = N/p^a$, we have

$$\begin{aligned} 2p^aL - 1 = \sigma(N) &= \sigma(p^a)\sigma(L) \\ &= (1 + p\sigma(p^{a-1}))\sigma(L) = \sigma(L) + p\sigma(p^{a-1})\sigma(L). \end{aligned}$$

Hence $p|\sigma(L) + 1$. If $p = \sigma(L) + 1$, then

$$\sum_{i=1}^{a+1} p^i = \sigma(p^a)p = \sigma(p^a)\sigma(L) + \sigma(p^a)$$

$$= \sigma(N) + \sigma(p^a) = 2p^aL - 1 + \sigma(p^a) = 2p^aL + \sum_{i=1}^{a} p^i,$$

or $p^{a+1} = 2p^aL$, showing that $N = 2^a$. Since N is OAP, $p \neq \sigma(L) + 1$, and so $p < \sigma(L)$ because $p|\sigma(L) + 1$. Then

$$\begin{aligned} \sigma(pN) &= \sigma(p^{a+1})\sigma(L) = (1 + p\sigma(p^a))\sigma(L) \\ &= \sigma(L) + p\sigma(N) = \sigma(L) + 2pN - p > 2pN, \end{aligned}$$

showing that pN is abundant.

Suppose M is a proper divisor of pN. If $p^{a+1} \nmid M$, then M is a divisor of N, and M is deficient because

$$S(M) \leqslant S(N) = 2 - 1/N < 2.$$

Suppose $p^{a+1} | M$. Then for some k, $p_k^{a_k} \nmid M$. Letting $q = p_k$ and $b = a_k$, we have $\sigma(p^a) \geqslant \sigma(q^b)$, or

$$\sum_{i=1}^{b} q^i \leqslant \sum_{i=1}^{a} p^i < \sum_{i=1}^{a+1} p^i.$$

Hence

$$(1/p^{a+1}) \sum_{i=0}^{b-1} q^{-i} < (1/q^b) \sum_{i=0}^{a} p^{-i},$$

and by adding $\Sigma_{i=0}^{a} p^{-i} \Sigma_{i=0}^{b-1} q^{-i}$ to both sides we obtain

$$\sum_{i=0}^{a+1} p^{-i} \sum_{i=0}^{b-1} q^{-i} < \sum_{i=0}^{a} p^{-i} \sum_{i=0}^{b} q^{-i},$$

or $S(p^{a+1})S(q^{b-1}) < S(p^a)S(q^b)$. Then

$$S(M) \leqslant S(p^{a+1})S(q^{b-1}) \prod_{i \neq j,k} S(p_i^{a_i})$$

$$< S(p^a)S(q^b) \prod_{i \neq j,k} S(p_i^{a_i}) = S(N) < 2,$$

showing that M is deficient. Q.E.D.

LEMMA 8. *If $N = \Pi_{i=1}^{r} p_i^{a_i}$ is OAP, a_i is even. If $p_1 = 3$, $a_1 \geqslant 12$.*

Proof. Suppose N is OAP, p^a is a component of N, q is a prime and $q | \sigma(p^a)$. Since $\sigma(N) = 2N - 1$ is odd and $\sigma(p^a) | \sigma(N)$, $\sigma(p^a) = \Sigma_{j=0}^{a} p^j$ is odd. Hence a is even. Since $q | 2\sigma(N) = 4N - 2$ and $4N$ is a perfect square, $(2|q) = 1$, where $(2|q)$ is the Legendre symbol, and so $q \equiv 1$ or $7 \pmod 8$ because $(2|q) = (-1)^{(q^2-1)/8}$. Also $\sigma(p^a) \equiv 1$ or $7 \pmod 8$, for, otherwise, $\sigma(p^a)$ would have a prime factor $\equiv 3$ or $5 \pmod 8$.

Suppose $p = 3$ and $a = 2e$. Then $\sigma(3^{2e}) \equiv 1 + 4e \equiv 1$ or $7 \pmod 8$, or $e \equiv 0 \pmod 2$. Hence $a = 4, 8, 12, \ldots$; however, $a \neq 4$ or 8 because $11 | \sigma(3^4)$, $11 \equiv 3 \pmod 8$, $13 | \sigma(3^8)$ and $13 \equiv 5 \pmod 8$. Q.E.D.

PROPOSITION 4. *If N is OAP, $\omega(N) \geqslant 6$.*

Proof. Suppose $N = \Pi_{i=1}^{r}$ is OAP. Then by Lemma 7 pN is primitive abundant for some $p | N$. If $3 \nmid N$, $\omega(N) \geqslant 7$, for, otherwise,

$$2 < S(pN) < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \leqslant \frac{5}{4}\frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\frac{19}{18} < 2.$$

Suppose $3 | N$. Then $3^{12} | pN$ by Lemma 8. According to the table of odd primitive abundant numbers M with fewer than five distinct prime factors in [4] $3^{12} \nmid M$.

Hence $\omega(N) \geqslant 5$, and $N \geqslant 3^{12}5^2 7^2 11^2 13^2 > 10^{13}$. Then $2 > S(N) = 2 - 1/N > 2 - 10^{-13}$, and by Lemma 6 $\omega(N) \geqslant 6$. Q.E.D.

For other results on QP and OAP see [3], [5], [6], [7] and [8].

Computer time for Tables 1 and 2 was over four hours.

TABLE 1

$N = \Pi_{i=1}^{5} p_i^{a_i}$ *for which* $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$ (a)

$p_1^{b_1}$	$p_2^{b_2}$	$p_3^{b_3}$	$p_4^{b_4}$	$p_5^{b_5}$
3^{25}	5^5	17^7	251	570407 (b)
3^{23}	5^{12}	17^6	257^4	65521 (c)
3^{22}	5^5	17^6	251	569659^2
3^{21}	5^9	17^9	257^4	65099^2 (b)
	5^5	17^5	251	557273
3^{20}	5^{14}	17^5	257^4	65357 (b)
3^{19}	5^3	17^3	181	57149^2
3^{18}	5^5	17^5	251	557017^2
		17^4	251	406811^2
3^{16}	5^5	17^8	251	567943^2
3^{12}	5^5	17^5	251	412943^2
3^{11}	5^{12}	17^9	257^3	58337 (c)
3^{10}	5^{10}	17^9	257^3	47791^2 (c)
3^9	7^3	13^5	19^2	1009643 (b)
3^8	5^{16}	17^8	257^4	15137^2 (c)
	5^{14}	17^3	251	1884527 (c)
				1884529
	5^{13}	17^3	251	1884061 (c)
	5^{11}	17^3	251	1870207
	5^9	17^3	251	1579769
	5^8	17^9	269^4	4153^3 (d)
	5^3	19^9	83^6	493277
		19^8	83^3	488203^2
		19^7	83^4	493201
3^7	5^6	17^2	233	(e)

Note: (a) If $b_i = a(p_i)$ and $c > 0$, Np_i^c also satisfies (1). See Lemma 1(a).
(b) See Lemma 1(b). (c) See Lemma 1(c). (d) See Lemma 1(d).
(e) $36549767 < p_5 < 36551083$.

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