

## Odd Integers $N$ With Five Distinct Prime Factors for Which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$

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**Abstract.** We make a table of odd integers  $N$  with five distinct prime factors for which  $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$ , and show that for such  $N$   $|\sigma(N)/N - 2| > 10^{-14}$ . Using this inequality, we prove that there are no odd perfect numbers, no quasiperfect numbers and no odd almost perfect numbers with five distinct prime factors. We also make a table of odd primitive abundant numbers  $N$  with five distinct prime factors for which  $2 < \sigma(N)/N < 2 + 2/10^{10}$ .

1. A positive integer  $N$  is called perfect, quasiperfect (QP), or almost perfect according as  $\sigma(N) = 2N$ ,  $2N + 1$ , or  $2N - 1$ , respectively, where  $\sigma(N)$  is the sum of the positive divisors of  $N$ . While twenty-four even perfect numbers are known, no odd perfect (OP) numbers, no QP numbers, and no almost perfect numbers except a power of 2 are known.

In this paper we make a table of odd integers  $N$  with five distinct prime factors for which

$$(1) \quad 2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12},$$

and we show that for such  $N$

$$|\sigma(N)/N - 2| > 10^{-14}.$$

Using this inequality, we prove that there are no OP, QP, or odd almost perfect (OAP) numbers with five distinct prime factors.

$N$  is called primitive abundant if  $N$  is abundant ( $\sigma(N) > 2N$ ) and every proper divisor  $M$  of  $N$  is deficient ( $\sigma(M) < 2M$ ). In 1913 Dickson [4] published a table of odd primitive abundant numbers with less than five distinct prime factors. In this paper we also make a table of odd primitive abundant numbers  $N$  with five distinct prime factors for which

$$(2) \quad 2 < \sigma(N)/N < 2 + 2/10^{10}.$$

2. Throughout this paper we let  $N = \prod_{i=1}^r p_i^{a_i}$  where  $3 \leq p_1 < \dots < p_r$  are primes and  $a_i$ 's are positive integers.  $p_i^{a_i}$  is called a component of  $N$ .

We define

$$\begin{aligned} a(p) &= \min\{a | p^{a+1} > 10^{12}\}, \\ \omega(N) &= r, \\ S(N) &= \sigma(N)/N = \prod_{i=1}^r (p_i^{a_i+1} - 1)/p_i^{a_i}(p_i - 1), \end{aligned}$$

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$$A(N) = \left[ \prod_{a_i < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_i \geq a(p_i)} S(p_i^{a(p_i)}) \right],$$

$$B(N) = \left[ \prod_{a_i < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_i \geq a(p_i)} p_i / (p_i - 1) \right],$$

$$L(p^a) = \begin{cases} [10^{12} \log S(p^a)] / 10^{12} & \text{if } a < a(p), \\ [10^{12} \log p / (p - 1)] / 10^{12} & \text{if } a \geq a(p), \end{cases}$$

where  $[ \ ]$  is the greatest integer function. We note that if  $p, q$  are primes with  $p > q$  and  $a, b$  are positive integers then

$$S(p^a) = (p^{a+1} - 1) / p^a (p - 1) < p / (p - 1) = \lim_{a \rightarrow \infty} S(p^a) \leq (q + 1) / q \leq S(q^b),$$

and so  $L(p^a) \leq L(q^b)$  and  $A(N) \leq S(N) \leq B(N)$ . Hence, we have

LEMMA 1. (a) If  $A(N) > 2 - 10^{-12}$  and  $B(N) < 2 + 10^{-12}$ ,  $N$  satisfies (1).

(b) If  $A(N) \leq 2 - 10^{-12} < B(N) < 2 + 10^{-12}$ , some  $N$  satisfies (1).

(c) If  $2 - 10^{-12} < A(N) < 2 + 10^{-12} \leq B(N)$ , some  $N$  satisfies (1).

(d) If  $A(N) < 2 - 10^{-12}$  and  $2 + 10^{-12} < B(N)$ , some  $N$  may satisfy (1).

(e) If  $2 + 10^{-12} < A(N)$  or  $B(N) < 2 - 10^{-12}$ ,  $N$  does not satisfy (1).

In Lemmas 2 through 5 we assume that  $N$  satisfies (1) and  $\omega(N) = 5$ .

LEMMA 2.

$$(3) \quad 0.6931471805544 < \sum_{i=1}^5 L(p_i^{b_i}) < 0.6931471805655,$$

where  $b_i = \min \{ a_i, a(p_i) \}$ .

*Proof.* Suppose  $p^a$  is a component of  $N$ . If  $a < a(p)$ , then

$$|\log S(p^a) - L(p^a)| < 10^{-12}.$$

If  $a \geq a(p)$ , then  $p^{a+1} > 10^{12}$  and

$$\begin{aligned} 10^{-12} &> \log p / (p - 1) - L(p^a) > \log S(p^a) - L(p^a) \geq \log S(p^a) - \log p / (p - 1) \\ &= \log (1 - 1/p^{a+1}) = - \sum_{i=1}^{\infty} 1/i(p^{a+1})^i > -1/(p^{a+1} - 1) \geq -10^{-12}. \end{aligned}$$

Hence

$$|\log S(p^a) - L(p^a)| < 10^{-12}.$$

Since (1) holds,

$$\begin{aligned} 0.6931471805544 &< \log(2 - 10^{-12}) - 5/10^{12} \\ &< \sum_{i=1}^5 \log S(p_i^{a_i}) - 5/10^{12} < \sum_{i=1}^5 L(p_i^{b_i}) \\ &< \sum_{i=1}^5 \log S(p_i^{a_i}) + 5/10^{12} < \log(2 + 10^{-12}) + 5/10^{12} \\ &< 0.6931471805655. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 3.  $p_1 = 3, p_2 \leq 11$  and  $p_3 \leq 41$ .

*Proof.* Lemma 3 follows from the following inequalities:

$$\frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} < 2 - 10^{-12},$$

$$\frac{3}{2} \frac{13}{12} \frac{17}{16} \frac{19}{18} \frac{23}{22} < 2 - 10^{-12},$$

$$\frac{3}{2} \frac{5}{4} \frac{43}{42} \frac{47}{46} \frac{53}{52} < 2 - 10^{-12}. \quad \text{Q.E.D.}$$

LEMMA 4.  $p_4 < 5000$ .

*Proof.* Suppose  $N$  satisfies (1) and  $p_4 \geq 5003$ . Then

$$\begin{aligned} 0 \leq L(p_5^{b_5}) &\leq L(p_4^{b_4}) < \log S(p_4^{b_4}) + 10^{-12} \\ &< \log p_4 / (p_4 - 1) + 10^{-12} < 1 / (p_4 - 1) + 10^{-12} \\ &< 0.0002. \end{aligned}$$

Hence by (3)

$$(4) \quad 0.69274 < \sum_{i=1}^3 L(p_i^{b_i}) < 0.69315.$$

A computer (PDP11 at the University of Toledo) was used to find  $\prod_{i=1}^3 p_i^{b_i}$  satisfying (4), but there were none. Q.E.D.

Similarly, we can prove

LEMMA 5.  $p_5 < 3000000$ , or  $\prod_{i=1}^4 p_i^{b_i} = 3^7 5^6 17^2 233$  and  $36549767 \leq p_5 \leq 36551083$ .

The computer was used to find  $N = \prod_{i=1}^5 p_i^{a_i}$  satisfying  $a_i \leq a(p_i)$ , Lemmas 3, 4, 5, and Lemma 2 or Lemma 1(b), (c), (d), with the result given in Table 1.

LEMMA 6. Suppose  $N = \prod_{i=1}^5 p_i^{a_i}$  and  $M = \prod_{i=1}^5 p_i^{b_i}$  where  $b_i = \min\{a_i, a(p_i)\}$ .

If  $M = 3^2 3^5 5^{12} 17^6 257^4 65521$ ,  $|S(N) - 2| > 5/10^{13}$ ;

if  $M = 3^8 5^{14} 17^3 251 \cdot 1884529$ ,  $|S(N) - 2| > 2/10^{14}$ ;

if  $M = 3^8 5^9 17^3 251 \cdot 1579769$ ,  $|S(N) - 2| > 3/10^{13}$ ;

if  $M = 3^8 5^8 17^9 269^4 4153^3$ ,  $|S(N) - 2| > 4/10^{14}$ ;

if  $\prod_{i=1}^4 p_i^{b_i} = 3^7 5^6 17^2 233$ ,  $|S(N) - 2| > 10^{-14}$ .

In all other cases  $|S(N) - 2| > 10^{-13}$ .

*Proof.* The first part of Lemma 6 follows from the following inequalities:

$$S(3^2 3^5 5^{12} 17^6 257^4 65521) < 2 - 5/10^{13},$$

$$S(3^2 3^5 5^{12} 17^6 257^5 65521) > 2 + 1/10^{12},$$

$$S(3^8 5^{14} 17^3 251) 1884529/1884528 < 2 - 2/10^{14},$$

$$S(3^8 5^9 17^3 251 \cdot 1579769) < 2 - 4/10^{13},$$

$$S(3^8 5^9 17^3 251 \cdot 1579769^2) > 2 + 3/10^{13},$$

$$S(3^8 5^8 269^4) 17/16 \cdot 4153/4152 < 2 - 4/10^{14},$$

$$S(3^8 5^8 17^9 269^5 4153^3) > 2 + 3/10^{13},$$

$$S(3^7 5^6 17^2 233 \cdot 36550379) > 2 + 5/10^{14},$$

$$S(3^7 5^6 17^2 233) 36550429/36550428 < 2 - 10^{-14}.$$

Suppose  $|S(N) - 2| \leq 10^{-13}$ . Then (1) holds, and so  $N$  is given in Table 1; however, for every  $N$  in Table 1 except for those given above  $S(N) \leq B(N) < 2 - 10^{-13}$ , or  $S(N) \geq A(N) > 2 + 10^{-13}$ . Q.E.D.

We have proved

**THEOREM.** *If  $N$  is an odd integer with  $\omega(N) = 5$ ,  $|\sigma(N)/N - 2| > 10^{-14}$ .*

3. We used a similar method to find odd primitive abundant numbers  $N = \prod_{i=1}^5 p_i^{a_i}$  for which (2) holds, with the result given in Table 2 in the microfiche. Table 2 includes odd primitive abundant numbers  $N$  with  $\omega(N) = 5$  one of whose component  $p^a$  is greater than  $10^{10}$ ; for, letting  $M = N/p^a$ , we have

$$\begin{aligned} 2 < \sigma(N)/N &= \sigma(M)\sigma(p^a)/Mp^a = \sigma(M)(p\sigma(p^{a-1}) + 1)/Mp^a \\ &= \sigma(Mp^{a-1})/Mp^{a-1} + \sigma(M)/Mp^a < 2 + 2/10^{10}, \end{aligned}$$

showing that (2) holds.

4. Suppose  $N$  is an odd integer such that  $\sigma(N) = 2N + A$ . If  $|A/N| \leq 10^{-14}$ , then by our Theorem  $\omega(N) \geq 6$ . We give three examples of such  $N$ .

Suppose  $N$  is OP. Sylvester (1888), Dickson (1913), and Kanold (1949) proved that  $\omega(N) \geq 5$ . From our Theorem we have

**PROPOSITION 1.** *If  $N$  is OP,  $\omega(N) \geq 6$ .*

This fact was also proved by Gradštein (1925), Kühnel (1949) and Webber (1951). Pomerance [1] (1972) and Robbins (1972) proved that  $\omega(N) \geq 7$ , and Hagis [2] proved that  $\omega(N) \geq 8$ .

**PROPOSITION 2.** *If  $N$  is QP,  $\omega(N) \geq 6$ .*

*Proof.* By [3] if  $N$  is QP, then  $N$  is an odd perfect square,  $\omega(N) \geq 5$  and  $N > 10^{20}$ . Hence  $2 < S(N) = 2 + 1/N < 2 + 10^{-20}$ , and so by Theorem  $\omega(N) \geq 6$ . Q.E.D.

**LEMMA 7.** *If  $N$  is OAP,  $pN$  is primitive abundant for some  $p|N$ .*

*Proof.* Suppose  $N = \prod_{i=1}^r p_i^{a_i}$  is OAP, and choose  $j$  so that  $\sigma(p_j^{a_j}) \geq \sigma(p_i^{a_i})$  for every  $i$ . Letting  $p = p_j$ ,  $a = a_j$  and  $L = N/p^a$ , we have

$$\begin{aligned} 2p^a L - 1 &= \sigma(N) = \sigma(p^a)\sigma(L) \\ &= (1 + p\sigma(p^{a-1}))\sigma(L) = \sigma(L) + p\sigma(p^{a-1})\sigma(L). \end{aligned}$$

Hence  $p|\sigma(L) + 1$ . If  $p = \sigma(L) + 1$ , then

$$\begin{aligned} \sum_{i=1}^{a+1} p^i &= \sigma(p^a)p = \sigma(p^a)\sigma(L) + \sigma(p^a) \\ &= \sigma(N) + \sigma(p^a) = 2p^a L - 1 + \sigma(p^a) = 2p^a L + \sum_{i=1}^a p^i, \end{aligned}$$

or  $p^{a+1} = 2p^a L$ , showing that  $N = 2^a$ . Since  $N$  is OAP,  $p \neq \sigma(L) + 1$ , and so  $p < \sigma(L)$  because  $p|\sigma(L) + 1$ . Then

$$\begin{aligned} \sigma(pN) &= \sigma(p^{a+1})\sigma(L) = (1 + p\sigma(p^a))\sigma(L) \\ &= \sigma(L) + p\sigma(N) = \sigma(L) + 2pN - p > 2pN, \end{aligned}$$

showing that  $pN$  is abundant.

Suppose  $M$  is a proper divisor of  $pN$ . If  $p^{a+1} \nmid M$ , then  $M$  is a divisor of  $N$ , and  $M$  is deficient because

$$S(M) \leq S(N) = 2 - 1/N < 2.$$

Suppose  $p^{a+1} \mid M$ . Then for some  $k$ ,  $p_k^{a_k} \nmid M$ . Letting  $q = p_k$  and  $b = a_k$ , we have  $\sigma(p^a) \geq \sigma(q^b)$ , or

$$\sum_{i=1}^b q^i \leq \sum_{i=1}^a p^i < \sum_{i=1}^{a+1} p^i.$$

Hence

$$(1/p^{a+1}) \sum_{i=0}^{b-1} q^{-i} < (1/q^b) \sum_{i=0}^a p^{-i},$$

and by adding  $\sum_{i=0}^a p^{-i} \sum_{i=0}^{b-1} q^{-i}$  to both sides we obtain

$$\sum_{i=0}^{a+1} p^{-i} \sum_{i=0}^{b-1} q^{-i} < \sum_{i=0}^a p^{-i} \sum_{i=0}^b q^{-i},$$

or  $S(p^{a+1})S(q^{b-1}) < S(p^a)S(q^b)$ . Then

$$\begin{aligned} S(M) &\leq S(p^{a+1})S(q^{b-1}) \prod_{i \neq j, k} S(p_i^{a_i}) \\ &< S(p^a)S(q^b) \prod_{i \neq j, k} S(p_i^{a_i}) = S(N) < 2, \end{aligned}$$

showing that  $M$  is deficient. Q.E.D.

**LEMMA 8.** *If  $N = \prod_{i=1}^r p_i^{a_i}$  is OAP,  $a_i$  is even. If  $p_1 = 3$ ,  $a_1 \geq 12$ .*

*Proof.* Suppose  $N$  is OAP,  $p^a$  is a component of  $N$ ,  $q$  is a prime and  $q \mid \sigma(p^a)$ . Since  $\sigma(N) = 2N - 1$  is odd and  $\sigma(p^a) \mid \sigma(N)$ ,  $\sigma(p^a) = \sum_{j=0}^a p^j$  is odd. Hence  $a$  is even. Since  $q \mid 2\sigma(N) = 4N - 2$  and  $4N$  is a perfect square,  $(2 \mid q) = 1$ , where  $(2 \mid q)$  is the Legendre symbol, and so  $q \equiv 1$  or  $7 \pmod{8}$  because  $(2 \mid q) = (-1)^{(q^2-1)/8}$ . Also  $\sigma(p^a) \equiv 1$  or  $7 \pmod{8}$ , for, otherwise,  $\sigma(p^a)$  would have a prime factor  $\equiv 3$  or  $5 \pmod{8}$ .

Suppose  $p = 3$  and  $a = 2e$ . Then  $\sigma(3^{2e}) \equiv 1 + 4e \equiv 1$  or  $7 \pmod{8}$ , or  $e \equiv 0 \pmod{2}$ . Hence  $a = 4, 8, 12, \dots$ ; however,  $a \neq 4$  or  $8$  because  $11 \mid \sigma(3^4)$ ,  $11 \equiv 3 \pmod{8}$ ,  $13 \mid \sigma(3^8)$  and  $13 \equiv 5 \pmod{8}$ . Q.E.D.

**PROPOSITION 4.** *If  $N$  is OAP,  $\omega(N) \geq 6$ .*

*Proof.* Suppose  $N = \prod_{i=1}^r p_i^{a_i}$  is OAP. Then by Lemma 7  $pN$  is primitive abundant for some  $p \mid N$ . If  $3 \nmid N$ ,  $\omega(N) \geq 7$ , for, otherwise,

$$2 < S(pN) < \prod_{i=1}^r \frac{p_i}{p_i - 1} \leq \frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} \frac{19}{18} < 2.$$

Suppose  $3 \mid N$ . Then  $3^{12} \mid pN$  by Lemma 8. According to the table of odd primitive abundant numbers  $M$  with fewer than five distinct prime factors in [4]  $3^{12} \nmid M$ .

Hence  $\omega(N) \geq 5$ , and  $N \geq 3^{12}5^27^211^213^2 > 10^{13}$ . Then  $2 > S(N) = 2 - 1/N > 2 - 10^{-13}$ , and by Lemma 6  $\omega(N) \geq 6$ . Q.E.D.

For other results on QP and OAP see [3], [5], [6], [7] and [8].

Computer time for Tables 1 and 2 was over four hours.

TABLE 1

$$N = \prod_{i=1}^5 p_i^{a_i} \text{ for which } 2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12} \text{ (a)}$$

$p_1^{b_1}$	$p_2^{b_2}$	$p_3^{b_3}$	$p_4^{b_4}$	$p_5^{b_5}$
$3^{25}$	$5^5$	$17^7$	251	570407 <sup>(b)</sup>
$3^{23}$	$5^{12}$	$17^6$	$257^4$	65521 <sup>(c)</sup>
$3^{22}$	$5^5$	$17^6$	251	$569659^2$
$3^{21}$	$5^9$	$17^9$	$257^4$	$65099^2$ <sup>(b)</sup>
	$5^5$	$17^5$	251	557273
$3^{20}$	$5^{14}$	$17^5$	$257^4$	$65357$ <sup>(b)</sup>
$3^{19}$	$5^3$	$17^3$	181	$57149^2$
$3^{18}$	$5^5$	$17^5$	251	$557017^2$
		$17^4$	251	$406811^2$
$3^{16}$	$5^5$	$17^8$	251	$567943^2$
$3^{12}$	$5^5$	$17^5$	251	$412943^2$
$3^{11}$	$5^{12}$	$17^9$	$257^3$	$58337$ <sup>(c)</sup>
$3^{10}$	$5^{10}$	$17^9$	$257^3$	$47791^2$ <sup>(c)</sup>
$3^9$	$7^3$	$13^5$	$19^2$	$1009643$ <sup>(b)</sup>
$3^8$	$5^{16}$	$17^8$	$257^4$	$15137^2$ <sup>(c)</sup>
	$5^{14}$	$17^3$	251	$1884527$ <sup>(c)</sup>
				1884529
	$5^{13}$	$17^3$	251	$1884061$ <sup>(c)</sup>
	$5^{11}$	$17^3$	251	1870207
	$5^9$	$17^3$	251	1579769
	$5^8$	$17^9$	$269^4$	$4153^3$ <sup>(d)</sup>
	$5^3$	$19^9$	$83^6$	493277
		$19^8$	$83^3$	$488203^2$
		$19^7$	$83^4$	493201
$3^7$	$5^6$	$17^2$	233	(e)

Note: (a) If  $b_i = a(p_i)$  and  $c > 0$ ,  $Np_i^c$  also satisfies (1). See Lemma 1(a).

(b) See Lemma 1(b). (c) See Lemma 1(c). (d) See Lemma 1(d).

(e)  $36549767 < p_5 < 36551083$ .

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1. C. POMERANCE, "Odd perfect numbers are divisible by at least seven distinct primes," *Acta Arith.*, v. 25, 1974, pp. 265–299.
2. P. HAGIS, JR., "Every odd perfect number has at least eight prime factors," Abstract #720-10-14, *Notices Amer. Math. Soc.*, v. 22, 1975, p. A-60.
3. H. L. ABBOT, C. E. AULL, E. BROWN & D. SURYANARAYANA, "Quasiperfect numbers," *Acta Arith.*, v. 22, 1973, pp. 439–447.  
H. L. ABBOT, C. E. AULL, E. BROWN & D. SURYANARAYANA, Corrections to the paper "Quasiperfect numbers," *Acta Arith.*, v. 29, 1976, pp. 427–428.
4. L. E. DICKSON, "Finiteness of the odd perfect and primitive abundant numbers with  $n$  distinct prime factors," *Amer. J. Math.*, v. 35, 1913, pp. 413–422.
5. M. KISHORE, "Quasiperfect numbers are divisible by at least six distinct prime factors," Abstract #75T-A113, *Notices Amer. Math. Soc.*, v. 22, 1975, p. A-441.
6. M. KISHORE, "Odd almost perfect numbers," Abstract #75T-A92, *Notices Amer. Math. Soc.*, v. 22, 1975, p. A-380.
7. R. P. JERRARD & N. TEMPERLEY, "Almost perfect numbers," *Math. Mag.*, v. 46, 1973, pp. 84–87.
8. J. T. CROSS, "A note on almost perfect numbers," *Math. Mag.*, v. 47, 1974, pp. 230–231.