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Some Extremal 2-Bases *

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Abstract. By means of a computer search, some extremal additive bases have been constructed which have heretofore been unknown.

A set $A : 0 = a_1 < a_2 < \cdots < a_k$ of integers is called a *2-basis for n* if each of $0, 1, 2, \ldots, n$ can be represented as the sum of two summands from A, with repetition of summands allowed. If the elements of A do not exceed $n/2$, we call A a *restricted* 2-basis for n. A more general idea is that of an h-basis, where h summands are used in the representation. However, in this note we shall be dealing only with 2-bases, which we may refer to more briefly as "bases". Rohrbach [5] considered the function $n(k) = n_2(k)$, which is the largest integer n for which a basis of k elements exists. An *extremal basis* is a basis of k elements for $n(k)$. Our object here is to provide a list of extremal bases for $2 \leqslant k \leqslant 14$, most of which have been unknown heretofore. We shall also compare the results of some basis constructions made by Rohrbach.

First we give some facts about $n(k)$. By a simple combinatorial argument one obtains

$$n(k) \leqslant \binom{k+1}{2}.$$

Rohrbach, by considering the distribution of the k basis elements, in a rather involved argument, improved this to

$$n(k) < .4992\, k^2,$$

this holding for sufficiently large k. By means of analytical arguments, Moser, Pounder, and Riddell [2], [3], [4] obtained improvements on this estimate, and the most recent improvement is that of Klotz [1], who combined Moser's and Rohrbach's methods to obtain

$$n(k) < .4802\, k^2.$$

On the other hand, Rohrbach (op. cit., p. 5) constructed a basis of k elements $(k > 1)$ for

$$(1) \qquad n = \frac{k^2}{4} + \frac{3}{2}k - \gamma,$$

where $\gamma = 2, 7/4, 2,$ or $11/4$ according as $k \equiv 0, 1, 2,$ or $3 \pmod 4$, and so

$$n(k) > k^2/4.$$

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Rohrbach conjectured (p. 9) that this estimate is the right size for $n(k)$; namely, he conjectured

$$n(k) = k^2/4 + O(k). \tag{2}$$

The attempt to prove this conjecture provided a large part of the motivation for the above-mentioned work on lowering the upper estimate for $n(k)$.

Rohrbach also had a second construction (p. 7), somewhat more complicated than the first one, which gave a slightly better result for larger k. For even $k \geqslant 12$, he constructed a basis for

$$n = k^2/4 + 2k - \delta, \tag{3}$$

where $\delta = 6$ or 7 according as $k \equiv 0$ or $2 \pmod 4$, while for odd $k \geqslant 43$ he obtained a basis for

$$n = \frac{k^2}{4} + \frac{11}{6}k - \delta' \tag{4}$$

where δ' depends on the least residue of k mod 12. In the following we shall denote by $R(k)$ the largest of (1), (3), and (4). Rohrbach's conjecture (2) evidently stemmed from a feeling that his constructions were almost as good as possible.

In the table below, all of the extremal bases for $2 \leqslant k \leqslant 14$ are listed. Those for $2 \leqslant k \leqslant 6$ were known before. Those for $k = 7$ and 8 were first found by means of a few days of hand labor [4]; later they were found in a few seconds with a computer. But for $k = 15$ the computing time would have been excessive, and so we stopped at $k = 14$. It is interesting to compare the results $R(k)$ of Rohrbach's constructions with $n(k)$, and these values are also listed in the table.

At the suggestion of the referee, we include some remarks on how the computer search was carried out. These are in the appendix.

Evidently, one would have to construct more extremal bases in order to see how to improve upon Rohrbach's constructions, since $R(k)$ is only beginning to fall behind $n(k)$ towards the end of the table. Rohrbach's types do not include the bases starting with 0, 1, 3, 4 (except for the basis 0, 1, 3, 4 itself); these may be the beginning of bases more efficient than his.

Rohrbach's constructed bases were restricted, and symmetric about $n/4$. That there is not always an *extremal* basis of this form for given k is illustrated by one case in the above table, namely by the case $k = 11$. In all other cases, at least one of the extremal bases is restricted and symmetric.

Stöhr [6], whose papers comprise a thorough review of facts about bases to 1955, wondered (p. 48) whether there might exist more than two extremal bases for any k. Our table shows that there do in some cases. It may be that the number of extremal bases for a given k is unbounded over $k = 2, 3, \ldots$.

Note that, in the table, all of the extremal bases that are restricted are also symmetric. We close with the question: If an extremal basis is restricted, is it necessarily symmetric?

k	Extremal bases	$n(k)$	$R(k)$
2	0, 1 (R_1)	2	2
3	0, 1, 2 (R_1); 0, 1, 3	4	4
4	0, 1, 3, 4 (R_1)	8	8
5	0, 1, 3, 5, 6 (R_1)	12	12
6	0, 1, 3, 5, 7, 8 (R_1)	16	16
7	0, 1, 2, 5, 8, 9, 10 (R_1)	20	20
	0, 1, 3, 5, 7, 9, 10 (R_1)		
	0, 1, 3, 4, 8, 9, 11		
	0, 1, 3, 4, 9, 11, 16		
	0, 1, 3, 5, 6, 13, 14		
8	0, 1, 2, 5, 8, 11, 12, 13 (R_1)	26	26
	0, 1, 3, 4, 9, 10, 12, 13		
	0, 1, 3, 5, 7, 8, 17, 18		
9	0, 1, 2, 5, 8, 11, 14, 15, 16 (R_1)	32	32
	0, 1, 3, 5, 7, 9, 10, 21, 22		
10	0, 1, 3, 4, 9, 11, 16, 17, 19, 20	40	38
11	0, 1, 2, 3, 7, 11, 15, 19, 21, 22, 24	46	44
	0, 1, 2, 5, 7, 11, 15, 19, 21, 22, 24		
12	0, 1, 3, 5, 6, 13, 14, 21, 22, 24, 26, 27 (R_2)	54	54
	0, 1, 3, 4, 9, 11, 16, 18, 23, 24, 26, 27		
	0, 1, 2, 3, 7, 11, 15, 19, 23, 25, 26, 28		
	0, 1, 2, 5, 7, 11, 15, 19, 23, 25, 26, 28		
13	0, 1, 3, 4, 9, 11, 16, 21, 23, 28, 29, 31, 32	64	60
14	0, 1, 3, 4, 9, 11, 16, 20, 25, 27, 32, 33, 35, 36	72	70

(The bases indicated by R_1 and R_2 are types constructible by Rohrbach's respective methods.)

Appendix. Basically, the algorithm consists of examining all combinations of k elements from $0, 1, 2, \ldots, m$ (for some m) to see which combinations are bases for m. We shall see later that many of the combinations can be eliminated easily. Before we try to find extremal bases for a new 'k' value, the extremal bases and $n(k)$ values for all lower k's must be known. When k is 2, 3, or 4, it is quite easy to work out the bases and $n(k)$ by hand, and these form a good starting point for a computerized algorithm. For demonstration, let us say that we want to find all the extremal bases with $k = 6$, while those with $k = 2, 3, 4$, and 5 are already known.

We pick a test-number m for $n(6)$ and try to find one or more bases for m comprised of 6 elements from the set $\{0, 1, 2, \ldots, m\}$. The choice of the first test-number m is made by a guess based on a combination of hand calculations and extrapolation from the sequence $n(2), n(3), n(4), n(5)$. If the test-number chosen is too high, so that no basis is found, it is lowered by 1 and the test is run again. If it is too low, so that likely several bases for it are found, each of these bases is examined to see how large a number it is a basis for, and the largest such number will be $n(6)$.

TABLE A

<table>
<tr><td></td><td colspan="5">$k = 6, \quad m = 16$</td></tr>
<tr><td>Set No.
Sets</td><td>1
(2)</td><td>2
(3, 4)</td><td>3
(5, 6, 7, 8)</td><td>4
(9, 10, 11, 12)</td><td>5
(13, 14, 15, 16)</td></tr>
<tr><td></td><td colspan="4">– – – – – – – – – – 3 – – – – – – – – – –</td><td>1</td></tr>
<tr><td></td><td colspan="3">– – – – – – – – – – 2 – – –</td><td>2</td><td>0</td></tr>
<tr><td></td><td colspan="2">– – – – 1 – –</td><td>2</td><td>1</td><td>0</td></tr>
<tr><td></td><td>0</td><td>2</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td></td><td>1</td><td>2</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td></td><td colspan="2">– – – – 1 – –</td><td>3</td><td>0</td><td>0</td></tr>
<tr><td></td><td>0</td><td>2</td><td>2</td><td>0</td><td>0</td></tr>
<tr><td></td><td>1</td><td>1</td><td>2</td><td>0</td><td>0</td></tr>
<tr><td></td><td>1</td><td>2</td><td>1</td><td>0</td><td>0</td></tr>
</table>

Distributions of elements from the five sets

Suppose a test-number m is chosen. We observe that the integers 0 and 1 must always appear in a basis. This cuts down the number of combinations to be tested. The integers $2, 3, \ldots, 12$ $(= n(5))$ are segmented into 4 sets, namely

$$(2), \quad (3, 4), \quad (5, 6, 7, 8), \quad (9, 10, 11, 12).$$

Here, the largest element in a set is $n(l)$, where l runs from 2 to 5. We also use the set

$$(13, 14, \ldots, m).$$

We cut down further the number of combinations to be examined by considering how the elements other than 0 and 1 can be distributed among the above 5 sets.

Among the 6 elements, two of which are 0 and 1, at least 3 must be chosen from the first 4 sets, since $n(5) = 12$. In the first step, exactly 3 elements are so chosen, leaving one element to be chosen from the fifth set. See Table A, where this distribution is represented by the first line

$$- - - - - - 3 - - - - - - - \quad 1.$$

There is a saving in noting that the 3 elements from the first 4 sets must be 3, 5, and 6, since $\{0, 1, 3, 5, 6\}$ is the only 5-element basis for 12. It is a simple matter to combine this basis of 5 elements with one element at a time from the fifth set and see whether the combination forms a basis for m. If a basis is found, it is printed out and the program tries the next combination. When these combinations are exhausted, we move to the next step.

In the second step, we choose no element from the fifth set and 4 elements from sets 1 to 4. Analogously to the first step, we choose the largest possible number of elements from the last of the sets involved, namely from set 4. At least 2 elements must be chosen from the first 3 sets, since $n(4) = 8$, and so at most 2 elements can be chosen from the fourth set. Hence in step 2 we have the distribution indicated in Table A by

$$- - - - - - 2 - - - \quad 2 \quad 0.$$

The two elements from the first 3 sets must be 3 and 4, since $\{0, 1, 3, 4\}$ is the only 4-element basis for 8. All remaining combinations with this distribution are tested, and the program then goes to the next step.

Each step involves a new distribution with as large a number of elements as possible chosen from the set with the highest number, then as large a number as possible from the next set down, and so on. In fact, from bottom to top, the rows in Table A are in lexicographical order from the right.

In the case $k = 6$, $m = 16$, only the one basis $\{0, 1, 3, 5, 7, 8\}$ is found. The program is run with test-number $m = 17$ to show that there is no 6-element basis for 17, and so $n(6) = 16$ and the above basis is extremal.

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