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A Method of Virtual Displacements for the Degenerate Discrete l_1 Approximation Problem

By W. Fraser and J. M. Bennett

Abstract. Given the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \ldots, m,$$

let $A_i = (a_{i1}, \ldots, a_{in})$. It is known that if the matrix $A = (a_{ij})$ has rank $k \leqslant n$, then there is a point X which provides a minimum of

$$R(X) = \sum_{i=1}^{m} |r_i(X)| = \sum_{i=1}^{m} |(A_i, X) - b_i|$$

such that $r_i(X) = 0$ for at least k values of the index i. If $r_i(X) = 0$ for exactly k values of the index i, the point or vertex is called ordinary, while if $r_i(X) = 0$ for more than k values of i, the vertex is termed degenerate.

A necessary and sufficient condition to determine if X minimizes R is valid if X is an ordinary vertex but not if X is degenerate. A degeneracy at X can be removed by applying perturbations to an appropriate number of the b_i so that X becomes an ordinary vertex of a modified problem. By noting that the test uses only values of the A_i, it is possible to avoid actual introduction of the perturbations to the b_i with a resulting substantial improvement of the efficiency of the computation.

1. Given the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, \ldots, m), \tag{1.1}$$

let $A_i = (a_{i1}, \ldots, a_{in})$. It is known that if the matrix $A = (a_{ij})$ has rank $k \leqslant n$, then there is a point X which provides a minimum of

$$R(X) = \sum_{i=1}^{m} |r_i(X)| = \sum_{i=1}^{m} |(A_i, X) - b_i| \tag{1.2}$$

such that $r_i(X) = 0$ for at least k values of the index i. The minimum value of R and a point X at which it occurs are usually found by replacing the problem with an equivalent linear programming problem in which a function is to be minimized subject to constraints, whereas the method to be described below solves the direct problem, namely to minimize a function free of constraints. This implies, for example, that the bookkeeping involved in setting up a linear program is not required for the application of this method.

If $r_i(X) = 0$ for exactly k values of i, X is referred to as an ordinary vertex,

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while if $r_i(X) = 0$ for more than k values of i, X is called degenerate. An edge is obtained by holding $r_i = 0$ for $(k - 1)$ values of i and allowing the remaining r_i to vary.

It is known that if X is an ordinary vertex, and $R(X)$ is nondecreasing for displacements along each of the edges emanating from X, then X is a minimum of R. On the other hand, if X is degenerate, R could be nondecreasing along edges generated by the subset of the system used to determine X, but still not be a minimum. Example 1 at the end of the article can be used to illustrate this point, using the first two equations of the system.

Degeneracy of a vertex can be removed by applying perturbations to the b_i for indices i corresponding to any $r_i(X) = 0$ not used to determine X. In this way the problem is transformed into one for which X is an ordinary vertex at which the test for sufficiency mentioned above is applicable. On a first coding the perturbations used to remove degeneracy were explicitly introduced in the program. The routine that resulted was found to be of more or less the same speed as the best routine known to us at the time [1] if the solution was found at an ordinary vertex. However, the computation required to cope with the perturbations in the degenerate case had the effect of slowing the routine seriously.

In this article a procedure at a degenerate vertex is described which is of a nature comparable to that which would occur if actual perturbations were introduced, but at the same time does not introduce them explicitly into the problem. By eliminating the arithmetic required previously, the resulting routine can be made significantly more efficient.

Except for minor modifications, the procedure to be described is applicable to a system with matrix of rank $k \leqslant n$, so that the discussion is restricted to the case of a system of rank n. The steps taken at an ordinary vertex are described in the following section so as to provide context for the discussion of the procedure at a degenerate vertex.

2. Let e_k denote the unit vector with 1 in position k and zero elsewhere, and let $\widetilde{A}$ denote the matrix of a subsystem of (1.1) consisting of n of the equations, say the first n, such that rank $\widetilde{A} = n$, and let $\widetilde{B}$ denote the vector (b_i), $i = 1, \ldots, n$. Edges which emanate from the vertex X which satisfies $\widetilde{A}X = \widetilde{B}$ have the directions of the vectors E_k which satisfy $\widetilde{A}E_k = e_k$, $k = 1, \ldots, n$.

If $r_i(X) = 0$ for exactly n values of i, a necessary and sufficient condition that $R(X)$ given by (1.2) have a minimum at X is that R should be nondecreasing for displacements from X in each of the directions E_k, $k = 1, \ldots, n$.

Suppose that X is an ordinary vertex, not necessarily a minimum, determined by solving the system $r_i(X) = 0$, $i = 1, \ldots, n$, and that $r_i(X) > 0$, $i = n + 1, \ldots, m$. Denoting the matrix of this sytem by $\widetilde{A}$, and column j of its inverse by C_j, points Y on edge j are given in terms of a parameter λ by

$$Y = X + \lambda C_j. \tag{2.1}$$

Also, it follows readily that for points Y on edge j we have in terms of λ

$$R(Y) = |\lambda| + \sum_{i=n+1}^{m} |r_i(X) + \lambda(A_i, C_j)|. \tag{2.2}$$

For all sufficiently small $|\lambda|$ this equation can be expressed in the form

$$R(Y) = |\lambda| + \sum_{i=n+1}^{m} [r_i(X) + \lambda(A_i, C_j)].$$

If $(A_i, C_j) = 0$, then $r_i(Y)$ remains constant along edge j. Otherwise, the zeros of $r_i(Y)$ are given by

$$\lambda_{ij} = -\frac{r_i(X)}{(A_i, C_j)} \qquad (i = n+1, \dots, m), (A_i, C_j) \neq 0. \tag{2.3}$$

R is a differentiable function of λ everywhere along the edge except at zeros of $r_i(Y)$, where the derivative jumps by the amount $2|(A_i, C_j)|$. Denoting the derivative of R on the left at $\lambda = 0$ by $R'(0-)$, and on the right by $R'(0+)$, we have

$$R'(0-) = -1 + \sum_{i=n+1}^{m} (A_i, C_j), \tag{2.4}$$

and

$$R'(0+) = 1 + \sum_{i=n+1}^{m} (A_i, C_j). \tag{2.5}$$

If one of $R'(0-)$ or $R'(0+)$ is zero, or if R' changes sign at $\lambda = 0$, then X is a minimum of R along edge j. If this is true for each edge j, then X is a minimum of R. If one or both of the derivatives at the minimum is zero, the minimum is not unique, while, if neither derivative at the minimum is zero, the minimum is unique.

If both $R'(0-)$ and $R'(0+)$ are negative, the positive λ_{ij} calculated by (2.3) are ordered according to increasing size, and to $R'(0+)$ we add successively terms $2|(A_i, C_j)|$ until the sum turns zero or positive. The index i for which this occurs, determines a vertex Y which produces a minimum of R on edge j. Equation j is replaced by equation i and the process repeated. A comparable procedure is followed if both $R'(0-)$ and $R'(0+)$ are positive at $\lambda = 0$.

In this process the only role played by the b_i is to attach a sign to $r_i(X)$, and the sufficiency test for a minimum involves only elements of the matrix A. This makes it possible to avoid explicit introduction of the perturbations to be described in the next section.

3. Suppose that the computation has led to a degenerate vertex X, found as the solution of $r_i(X) = 0$ $(i = 1, \dots, n)$, and that for a positive integer p, $0 < p < m - n$, we also have $r_i(X) = 0$ $(i = n+1, \dots, n+p)$, while $r_i(X) > 0$, $n + p < i \leqslant m$. In order to test X when it is degenerate, perturbations are applied as in (3.1) below, so that X becomes an ordinary vertex for a modified problem, and the sufficiency test for a minimum applies. The perturbations are made as follows:

For $\mu > 0$, $\eta_i > 0$, set

$$
\begin{aligned}
b_i' &= b_i - \mu\eta_i \qquad (i = n+1, \ldots, n+p) \\
&= b_i \qquad\qquad\; \text{(otherwise)}.
\end{aligned} \tag{3.1}
$$

Here μ is a parameter which we assume can be assigned arbitrarily small positive values, and the nonzero η_i are to be specified later. Denote the resulting perturbed function by $R_\mu(X)$.

With the modifications (3.1) the vertex X becomes an ordinary vertex for the system $R_\mu(X)$ such that at X each $r_i(X) > 0$ $(n < i \leqslant m)$. For Y given by (2.1) we have

$$
R_\mu(Y) = |\lambda| + \sum_{i=n+1}^{n+p} |\mu\eta_i + \lambda(A_i, C_j)| + \sum_{i=n+p+1}^{m} |r_i(X) + \lambda(A_i, C_j)|. \tag{3.2}
$$

As before, if $(A_i, C_j) = 0$, then $r_i(Y)$ remains constant along edge j. For those A_i such that $(A_i, C_j) \neq 0$, the zeros at which the derivative of R_μ has jumps in value are given by

$$
\begin{aligned}
\lambda_{ij} &= -\frac{\mu\eta_i}{(A_i, C_j)} \qquad (i = n+1, \ldots, n+p), (A_i, C_j) \neq 0, \\
&= -\frac{r_i(X)}{(A_i, C_j)} \qquad (i = n+p+1, \ldots, m), (A_i, C_j) \neq 0.
\end{aligned} \tag{3.3}
$$

All of the λ_{ij} from (3.3) for $i = n+1, \ldots, n+p$ can be made smaller in absolute value than the λ_{ij} corresponding to $i > n+p$ by choosing μ to be sufficiently small. Thus, if searching for a minimum along edge j, we can assume that hyperplanes from the set corresponding to $i = n+1, \ldots, n+p$ are met first. Furthermore, the relative sizes of the η_i can be adjusted so that these hyperplanes are met in any specified order. Thus, if the η_i are equal, the λ_{ij} are proportional to the reciprocals of the (A_i, C_j), and the hyperplanes are met in this reciprocal of magnitude order. It is also possible to choose the η_i so that the hyperplanes are met in order of increasing index i. Of course, the precaution has to be taken that if a search is to be in the direction of increasing positive λ, only those indices i for which $(A_i, C_j) < 0$ should be included, and similarly for a search in the direction of negative λ.

We have tended to favor the search according to increasing index i, both because it is convenient for programming, and we have been unable to prove that any other strategy is better. For the same reasons the search for a minimum along edge j has been conducted according to order of increasing index j. It is possible to use other criteria, such as finding an edge with the largest gradient, or maximizing the reduction in R per cycle of calculation, but we have not been able to establish any clear superiority.

In carrying out the computation we assume, without actually doing so, that the perturbations $\mu\eta_i$ have been applied for $i = n+1, \ldots, n+p$. The test for a minimum is now applied successively along each of the edges emanating from X. If the test is satisfied in each case, then X is a minimum of R. Otherwise a search is carried out along the first edge encountered for which X is not a minimum, with the direction of search being determined as described before. If the search along edge j is in the direction of increasing positive λ, no sign changes are made in equation j; but if hyperplane i is passed, the signs of b_i and the components of A_i are changed. If the test for a minimum is satisfied in a member of the set corresponding to $(n + 1) \leqslant i \leqslant (n + p)$, no sign changes are made in the equation at which the minimum is encountered, and an exchange of equations is made in the usual way. By carrying out the calculation in this way, the virtual vertex under consideration at any time is always on the positive side of any hyperplane on which it does not lie for $i = 1, \ldots, n + p$. Of course, in the calculation as performed, only an exchange of equation occurs without a change of vertex. If the search is in the direction of decreasing negative λ, a similar procedure is followed with the exception that in this case the signs of b_j and the components of A_j are changed. If after all terms corresponding to indices $(n + 1)$ to $(n + p)$ have been used, a minimum has not been reached, the remaining λ_{ij} given by (3.3) are ordered as described previously and the minimum on edge j determined accordingly. In this case the vertex is not a minimum of the original system and an exchange of equations is made in the usual way. The new vertex so determined is then tested according to whether it is ordinary or degenerate.

In the case of the previously cited method of applying perturbations which would be subsequently held fixed, it was possible for us to show that the testing of vertices of the set so generated must terminate in a finite number of cycles, either with the decision that one of them is a minimum, or by calling for an exchange to a vertex other than one of the set. This comes from the fact that exactly one function is involved, and an exchange of equations is not made unless the function actually decreases. However, the technique of virtual displacements which has been described is not exactly the same as this; for one thing it assumes that the relative sizes of perturbations may be altered as needed while the computation is in progress. We have not yet proved that for this process cycling within the subset of the system consisting of equations satisfied at the degenerate vertex cannot occur, but we have never encountered it, and we conjecture that it is unlikely that it will occur.

4. The two examples which follow illustrate the exchange technique; the first involves only degenerate vertices, while both ordinary and degenerate vertices are encountered in the second.

Example 1. Where ϵ is interpreted as a small positive number, find the l_1 solution of the system

(1) $(4 + \epsilon)x - y = 0$,

(2) $(4 - \epsilon)x - y = 0$,

(3) $y = 0$,

(4) $(8 + \epsilon)x - 2y - (8 + \epsilon) = 0$,

(5) $(8 - \epsilon)x - 2y - (8 - \epsilon) = 0$.

The procedure begins by finding the vertex determined by the first two equations. Calculations performed as described in the text are tabulated, and actions required for the next cycle of the calculation are described below the tabulations.

Cycle 1.

$$\text{Vertex } (0, 0);\ A = \begin{pmatrix} 4+\epsilon & -1 \\ 4-\epsilon & -1 \end{pmatrix};\quad A^{-1} = \begin{pmatrix} \frac{1}{2\epsilon} & -\frac{1}{2\epsilon} \\ \frac{4-\epsilon}{2\epsilon} & -\frac{4+\epsilon}{2\epsilon} \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)
(1)	$(4+\epsilon)x - y = 0$	0	1
(2)	$(4-\epsilon)x - y = 0$	0	0
(3)	$y = 0$	0	$\frac{4-\epsilon}{2\epsilon}$
(4)	$-(8+\epsilon)x + 2y + (8+\epsilon) = 0$	$8+\epsilon$	$-3/2$
(5)	$-(8-\epsilon)x + 2y + (8-\epsilon) = 0$	$8-\epsilon$	$-1/2$

$r_i(0, 0) = 0$ for three values of i, so that $(0, 0)$ is degenerate.

$\Sigma_{i=1}^5 (A_i, C_1) = 2/\epsilon - 3/2$. Thus, $R'(0-) = 2/\epsilon - 7/2 > 0$ and $R'(0+) = 2/\epsilon - 3/2 > 0$, for sufficiently small ϵ. A search for the minimum is made through the positive (A_i, C_1). Since

$$\left(\frac{2}{\epsilon} - \frac{7}{2}\right) - \left(\frac{4-\epsilon}{\epsilon}\right) = \frac{-4-5\epsilon}{2\epsilon} < 0,$$

the signs in Eq. (1) are changed and Eqs. (1) and (3) are interchanged.

Cycle 2.

$$\text{Vertex } (0, 0);\quad A = \begin{pmatrix} 0 & 1 \\ 4-\epsilon & -1 \end{pmatrix};\quad A^{-1} = \begin{pmatrix} \frac{1}{4-\epsilon} & \frac{1}{4-\epsilon} \\ 1 & 0 \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)	(A_i, C_2)
(1)	$y = 0$	0	1	0
(2)	$(4-\epsilon)x - y = 0$	0	0	1
(3)	$-(4+\epsilon)x + y = 0$	0	$-\frac{2\epsilon}{4-\epsilon}$	$-\frac{4+\epsilon}{4-\epsilon}$
(4)	$-(8+\epsilon)x + 2y + (8+\epsilon) = 0$	$8+\epsilon$	$-\frac{3\epsilon}{4-\epsilon}$	$-\frac{8+\epsilon}{4-\epsilon}$
(5)	$-(8-\epsilon)x + 2y + (8-\epsilon) = 0$	$8-\epsilon$	$-\frac{\epsilon}{4-\epsilon}$	$-\frac{8-\epsilon}{4-\epsilon}$

$\Sigma_{i=1}^5 (A_i, C_1) = (4 - 7\epsilon)/(4 - \epsilon)$, and the test for a minimum on this edge is satisfied.

$\Sigma_{i=1}^5 (A_i, C_2) = (-16 - 2\epsilon)/(4 - \epsilon)$. The test for a minimum is not satisfied. The relations

$$R'(0+) + 2|(A_3, C_2)| = -\frac{8}{4-\epsilon} < 0 \quad \text{and}$$

$$R'(0+) + 2|(A_3, C_2)| + 2|(A_4, C_2)| = \frac{8+2\epsilon}{4-\epsilon} > 0$$

imply that the signs in Eq. (3) are to be changed, and an interchange is made between (4) and (2). Since (4) and (5) are encountered simultaneously, the test is made according to order of increasing index.

Cycle 3.

$$\text{Vertex } (1, 0);\quad A = \begin{pmatrix} 0 & 1 \\ -8-\epsilon & 2 \end{pmatrix};\quad A^{-1} = \begin{pmatrix} \frac{2}{8+\epsilon} & -\frac{1}{8+\epsilon} \\ 1 & 0 \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)	(A_i, C_2)
(1)	$y = 0$	0	1	0
(2)	$-(8+\epsilon)x + 2y + (8+\epsilon) = 0$	0	0	1
(3)	$(4+\epsilon)x - y = 0$	$4+\epsilon$	$\frac{\epsilon}{8+\epsilon}$	$-\frac{4+\epsilon}{8+\epsilon}$
(4)	$(4-\epsilon)x - y = 0$	$4-\epsilon$	$-\frac{3\epsilon}{8+\epsilon}$	$-\frac{4-\epsilon}{8+\epsilon}$
(5)	$-(8-\epsilon)x + 2y + (8-\epsilon) = 0$	0	$\frac{4\epsilon}{8+\epsilon}$	$\frac{8-\epsilon}{8+\epsilon}$

$\Sigma_{i=1}^{5}(A_i, C_1) = (8+3\epsilon)/(8+\epsilon)$, so that the test for a minimum is satisfied along edge 1.

$\Sigma_{i=1}^{5}(A_i, C_2) = 8\epsilon/(8+\epsilon)$, so that the test for a minimum is also satisfied along edge 2.

Thus, R has a minimum value 8 which occurs at $(1, 0)$.

Example 2. Find the l_1 solution of the following system. The system is shown with signs adjusted so that at the initial vertex, nonzero residuals are positive.

Cycle 1.

$$\text{Vertex } (-1, 1);\quad A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix};\quad A^{-1} = \begin{pmatrix} \frac{5}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)	λ	Order of Encounter
(1)	$x - 2y + 3 = 0$	0	1		
(2)	$3x + 5y - 2 = 0$	0	0		
(3)	$-2x - y - 1 = 0$	0	$-7/11$		
(4)	$x - 3y + 4 = 0$	0	$14/11$		
(5)	$-x - y = 0$	0	$-2/11$		
(6)	$4x + y + 3 = 0$	0	$17/11$		

(7)	$-3x - 4y + 1 = 0$	0	$-3/11$		
(8)	$-2x + y + 1 = 0$	4	$-13/11$	3.38	
(9)	$-5x - y - 1 = 0$	3	$-22/11$	1.50	(3)
(10)	$-3x + y + 2 = 0$	6	$-18/11$	3.67	
(11)	$-4x + 3y + 1 = 0$	8	$-29/11$	3.03	
(12)	$-4x - y - 1 = 0$	2	$-17/11$	1.29	(2)
(13)	$-x + y = 0$	2	$-8/11$	2.75	(4)
(14)	$-3x + 2y + 1 = 0$	6	$-21/11$	3.14	
(15)	$-3x - y - 1 = 0$	1	$-12/11$	0.92	(1)
(16)	$x + y + 1 = 0$	1	$2/11$		
(17)	$-x + 2y = 0$	3	$-11/11$	3.00	
(18)	$-4x + y + 3 = 0$	8	$-23/11$	3.83	

$\Sigma_{i=1}^{18}(A_i, C_1) = -142/11$. The search through indices i for which $(A_i, C_1) < 0$ goes through indices 3, 5, 7, 15, 12, 9 and 13 before the derivative turns nonnegative. Signs in Eqs. (3), (5), (7), (9), (12), (15) are all changed, and Eqs. (1) and (13) are interchanged.

Cycle 2.

$$\text{Vertex } \left(\frac{1}{4}, \frac{1}{4}\right); \quad A = \begin{pmatrix} -1 & 1 \\ 3 & 5 \end{pmatrix}; \quad A^{-1} = \begin{pmatrix} -5/8 & 1/8 \\ 3/8 & 1/8 \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)	(A_i, C_2)	λ_{i2}	Order of Encounter
(1)	$-x + y = 0$	0	1	0		
(2)	$3x + 5y - 2 = 0$	0	0	1		
(3)	$2x + y + 1 = 0$	7/4	$-7/8$	3/8	4.67	
(4)	$x - 3y + 4 = 0$	7/2	$-14/8$	$-2/8$		
(5)	$x + y = 0$	1/2	$-2/8$	2/8	2.00	(2)
(6)	$4x + y + 3 = 0$	17/4	$-17/8$	5/8	6.80	
(7)	$3x + 4y - 1 = 0$	3/4	$-3/8$	7/8	.86	(1)
(8)	$-2x + y + 1 = 0$	3/4	13/8	$-1/8$		
(9)	$5x + y + 1 = 0$	5/2	$-22/8$	6/8	3.33	
(10)	$-3x + y + 2 = 0$	3/2	18/8	$-2/8$		
(11)	$-4x + 3y + 1 = 0$	3/4	29/8	$-1/8$		
(12)	$4x + y + 1 = 0$	9/4	$-17/8$	5/8	3.60	
(13)	$x - 2y + 3 = 0$	11/4	$-11/8$	$-1/8$		
(14)	$-3x + 2y + 1 = 0$	3/4	21/8	$-1/8$		
(15)	$3x + y + 1 = 0$	2	$-12/8$	4/8	4.00	
(16)	$x + y + 1 = 0$	3/2	$-2/8$	2/8	6.00	
(17)	$-x + 2y = 0$	1/4	11/8	1/8	2.00	(3)
(18)	$-4x + y + 3 = 0$	9/4	23/8	$-3/8$		

$\Sigma_{i=1}^{18}(A_i, C_1) = 0$, so that the test for a minimum along edge 1 is satisfied.

$\Sigma_{i=1}^{18}(A_i, C_2) = 4$, so that $R'(0-) = 2$ and $R'(0+) = 4$. The search for the minimum on edge 2 goes past equation with index (7) and the minimum is encountered at index (5). Signs of Eqs. (7) and (2) are changed, and (2) and (5) are interchanged.

Cycle 3.

$$\text{Vertex } (0, 0); \quad A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}; \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)
(1)	$-x + y = 0$	0	1
(2)	$x + y = 0$	0	0
(3)	$-x + 2y = 0$	0	3/2
(4)	$2x + y + 1 = 0$	1	$-1/2$
(5)	$x - 3y + 4 = 0$	4	$-4/2$
(6)	$-3x - 5y + 2 = 0$	2	$-2/2$
(7)	$4x + y + 3 = 0$	3	$-3/2$
(8)	$-3x - 4y + 1 = 0$	1	$-1/2$
(9)	$-2x + y + 1 = 0$	1	3/2
(10)	$5x + y + 1 = 0$	1	$-4/2$
(11)	$-3x + y + 2 = 0$	2	4/2
(12)	$-4x + 3y + 1 = 0$	1	7/2
(13)	$4x + y + 1 = 0$	1	$-3/2$
(14)	$x - 2y + 3 = 0$	3	$-3/2$
(15)	$-3x + 2y + 1 = 0$	1	5/2
(16)	$3x + y + 1 = 0$	1	$-2/2$
(17)	$x + y + 1 = 0$	1	0
(18)	$-4x + y + 3 = 0$	3	5/2

$\Sigma_{i=1}^{18}(A_i, C_1) = 3$, so that $R'(0-) = 1$, and $R'(0+) = 3$. The minimum is encountered on this edge at index 3. Thus, change signs of (1) and interchange (1) and (3).

Cycle 4.

$$\text{Vertex } (0, 0); \quad A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}; \quad A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Index	Equation	Residual	(A_i, C_1)	(A_i, C_2)
(1)	$-x + 2y = 0$	0	1	0
(2)	$x + y = 0$	0	0	1

(3)	$x - y = 0$	0	$-2/3$	$1/3$
(4)	$2x + y + 1 = 0$	1	$-1/3$	$5/3$
(5)	$x - 3y + 4 = 0$	4	$-4/3$	$-1/3$
(6)	$-3x - 5y + 2 = 0$	2	$-2/3$	$-11/3$
(7)	$4x + y + 3 = 0$	3	$-3/3$	$9/3$
(8)	$-3x - 4y + 1 = 0$	1	$-1/3$	$-10/3$
(9)	$-2x + y + 1 = 0$	1	$3/3$	$-3/3$
(10)	$5x + y + 1 = 0$	1	$-4/3$	$11/3$
(11)	$-3x + y + 2 = 0$	2	$4/3$	$-5/3$
(12)	$-4x + 3y + 1 = 0$	1	$7/3$	$-5/3$
(13)	$4x + y + 1 = 0$	1	$-3/3$	$9/3$
(14)	$x - 2y + 3 = 0$	3	$-3/3$	0
(15)	$-3x + 2y + 1 = 0$	1	$5/3$	$-4/3$
(16)	$3x + y + 1 = 0$	1	$-2/3$	$7/3$
(17)	$x + y + 1 = 0$	1	0	$3/3$
(18)	$-4x + y + 3 = 0$	3	$5/3$	$-7/3$

$\Sigma_{i=1}^{18}(A_i, C_1) = 2/3$, so that $R'(0-) = -4/3$ and $R'(0+) = 2/3$, and the test for a minimum is satisfied along edge 1.

$\Sigma_{i=1}^{18}(A_i, C_2) = 2/3$, so that the test for a minimum is also satisfied along edge 2.

Thus, the test for a minimum is satisfied along both edges; and furthermore, the minimum is unique. The unique minimum value of R is 26 and this occurs at $x = y = 0$.

Department of Mathematics and Statistics
University of Guelph
Guelph, Ontario, Canada

Department of Computer Science
University of Western Ontario
London, Ontario, Canada

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