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The Comparison of Numerical Methods for Solving Polynomial Equations

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Abstract. In this paper we compare the Turán process [5]–[6] with the Lehmer-Schur method [2]. We prove that the latter is better.

1. **The Algorithms.** We first describe the Turán process [5]–[6] which can be considered as an improvement of Graeffe's method. For the complex polynomial

$$p_0(z) \equiv \sum_{j=0}^{n} a_{j0} z^j = 0 \qquad (a_{j0} \in \mathbf{C},\ a_{00}a_{n0} \neq 0), \tag{1.1}$$

the method can be formulated as follows.

Let

$$p_j(z) \equiv p_{j-1}(\sqrt{z})p_{j-1}(-\sqrt{z}) \equiv \sum_{k=0}^{n} a_{kj} z^j \qquad (j = 1, 2, \ldots) \tag{1.2}$$

be the jth Graeffe transformation and let

$$M[p_0(z), m_0] = \left[\max_{1 \leqslant k \leqslant n} \left|\frac{\sigma_k}{n}\right|^{\mu_0/k}\right]^{-1}, \tag{1.3}$$

where $\mu_0 = 2^{-m_0}$, $\sigma_0 = 0$,

$$\sigma_k = \left[k a_{km_0} - \sum_{j=1}^{k-1} a_{jm_0}\sigma_{k-j}\right] \Big/ a_{0m_0} \qquad (k = 1, \ldots, n) \tag{1.4}$$

and $m_0 \geqslant 1$ is fixed.

Let the constants α_{m_0}, l be defined by the inequalities

$$0.5 < \alpha_{m_0} < 5^{-\mu_0}, \qquad l > \pi\left[\arccos \frac{2.5 + \alpha_{m_0}}{2 + 2\alpha_{m_0}}\right]^{-1} - 1, \qquad m_0 \geqslant 2.$$

Then with the notations

$$M^{(0)} = M[p_0(z), m_0], \qquad S^{(0)} = 0, \tag{1.5}$$

the dth step of the algorithm is the following:

1. *Algorithm* (*T*). (i) Let

$$S_j^{(d+1)} = S^{(d)} + 0.5(1 + \alpha_{m_0})M^{(d)}\exp\left(j\frac{2\pi i}{l+1}\right),$$

where $j = 0, 1, \ldots, l$ and $i = \sqrt{-1}$.

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(ii) If there exists an index j such that $p_0(S_j^{(d+1)}) = 0$, then we get a root and the process terminates.

(iii) Let us compute the quantities

$$M_j^{(d+1)} = M[p_0(z + S_j^{(d+1)}), m_0] \qquad (j = 0, 1, \ldots, l)$$

and let

$$M^{(d+1)} = \min_j M_j^{(d+1)} = M_{j(d)}^{(d+1)}, \qquad S^{(d+1)} = S_{j(d)}^{(d+1)}.$$

Turán [5] proved that $S^{(d)}$ tends to a root of $p_0(z)$, and the convergence is linear. Turán [5] also proved that the number of iterations needed to achieve an arbitrary relative error ϵ (> 0) is independent of $p_0(z)$ and depends on degree $p_0(z)$ only.

Our purpose is to answer the remarks of the last section of [6]. For this reason we compare the Turán process with the Lehmer-Schur method which is often applied in practice ([2], [3], [4]). This algorithm can be described as follows.

Let

$$T[p_0(z)] = \sum_{j=0}^{n-1} (\bar{a}_{00}a_{j0} - a_{n0}\bar{a}_{n-j,0})z^j \tag{1.6}$$

and

$$T^j[p_0(z)] = T\{T^{j-1}[p_0(z)]\} \qquad (j = 2, \ldots). \tag{1.7}$$

Let us compute the numbers $c_j = T^j[p_0(0)]$, $(j = 1, \ldots, k)$, where

$$k = \min\{m \in \mathbf{N} | c_m = 0\}. \tag{1.8}$$

Here, $\mathbf{N}$ denotes the set of nonnegative integers. With the aid of the sequence $\{c_j\}_{j=1}^k$ we define the function $N[p_0(z)]$ as follows

$$N[p_0(z)] = \begin{cases} 1 & \text{if } \exists j \in \{1, \ldots, k-1\} \text{ such that } c_j < 0, \\ 0 & \text{if } c_j > 0 \ (j = 1, \ldots, k-1) \text{ and degree } T^{k-1}[p_0(z)] = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Lehmer [2] proved that if $N[p_0(z)] = 1$ then the polynomial $p_0(z)$ has a root in $\{z \in \mathbf{C} | |z| \leqslant 1\}$, if $N[p_0(z)] = 0$ then $p_0(z)$ has no roots in this set. We shall return to the case $N[p_0(z)] = -1$.

Let us introduce the notations

$$\alpha_j^{(d)} = \begin{cases} 0.5\gamma_0^{(d)} R^{(d-1)} & (j = 0), \\ 0.4\gamma_j^{(d)} R^{(d-1)} & (j = 1, \ldots, 8), \end{cases} \tag{1.9}$$

and

$$\beta_j^{(d)} = \begin{cases} z^{(d-1)} & (j = 0), \\ z^{(d-1)} + \dfrac{0.75R^{(d-1)}}{\cos\frac{\pi}{8}} \exp\left(\dfrac{2\pi i(j-1)}{8}\right) & (j = 1, \ldots, 8), \end{cases} \tag{1.10}$$

where the sequences $\{R^{(d)}\}$, $\{z^{(d)}\}$ and $\{\gamma_j^{(d)}\}$ are defined by the dth step of the Lehmer-Schur method ($d = 1, \ldots$). Let $\widetilde{p}_0(z) = p_0(z)/\psi$ ($\psi > 0$) and

$$z^{(0)} = 0; \qquad R^{(0)} = 1 + \max_j \left| \frac{a_{j0}}{a_{n0}} \right| . \tag{1.11}$$

Then the dth step can be written as follows.

2. *Algorithm* (L). (i) If there exists an index j such that $p_0(\beta_j^{(d)}) = 0$, then we get a root and the process terminates.

(ii) We choose the index $j \in \{0, 1, \ldots, 8\}$ such that

$$N[\widetilde{p}_0(\alpha_j^{(d)} z + \beta_j^{(d)})] = 1$$

and let

$$z^{(d)} = \beta_j^{(d)}, \qquad R^{(d)} = \alpha_j^{(d)}.$$

The numbers $\gamma_j^{(d)} \in [1, 1 + \delta]$, ($\delta \leqslant 0.5$) are chosen such that $N[\widetilde{p}_0(\alpha_j^{(d)} z + p_j^{(d)})] \geqslant 0$ will be satisfied (except in unusual circumstances $\gamma_j^{(d)} = 1$ can be chosen). Lehmer [2] proved that process converges linearly. The number of iteration steps needed to achieve an arbitrary absolute error ϵ (> 0) depends on $p_0(z)$.

2. **The Limitations of the Algorithms.** Denote by $\mathbf{Z}$ the set of integers and let $\mathbf{P}_n$ be the set of complex polynomials of degree n.

A numerical method M (iterative process) for solving $p_0(z) = 0$ where $p_0(z) \in \mathbf{P}_n$ can be identified with the sequence $\{b_k\} \subset \mathbf{C}$ which rises from the computation. This sequence depends on $p_0(z)$ and will be denoted by $\{Mp_0\} = \{b_k\}$. There exists a subsequence $\{b_{k_j}\}$ of $\{b_k\}$ such that

$$z^* = \lim_{j \to \infty} b_{k_j} \quad \text{and} \quad p_0(z^*) = 0. \tag{2.1}$$

A digital computer can perform elementary (complex) operations only over the finite set

$$S[0, K] \cap \mathbf{C}_\delta, \tag{2.2}$$

where $S[0, K] = \{z \in \mathbf{C} \mid |z| \leqslant K\}$ and

$$C_\delta = \{z \in \mathbf{C} | z = k\delta + j\delta i \colon k, j \in \mathbf{Z}\} \quad (\delta > 0). \tag{2.3}$$

If there exists an element b_{k_0} in the sequence $\{b_k\}$ such that $|b_{k_0}| > K$, then the algorithm M cannot continue to run because of overflow.

In order to study the overflow we introduce the class of polynomials

$$\mathbf{P}_M(a, K, K^*) = \{p_0(z) \in \mathbf{P}(a, K^*) \mid \{Mp_0\} \subset S[0, K], |\{Mp_0\}| = \infty\}, \tag{2.4}$$

where

$$\mathbf{P}(a, K^*) = \{p_0(z) \in \mathbf{P}_n \mid 0 < |z_j| \leqslant a \ (j = 1, \ldots, n), \|p_0(z)\| \leqslant K^*\} \tag{2.5}$$

and

$$\|p_0(z)\| = \max_j |a_{j0}|. \tag{2.6}$$

Here $|\{Mp_0\}|$ denotes the cardinality of $\{b_k\}$, and z_j is the jth zero of $p_0(z)$.

The set $\mathbf{P}_M(a, K, K^*)$ represents the class of all polynomials which can be solved by M in a bounded set.

The following statements are valid.

THEOREM 2.1. *The set* $\mathbf{P}_T(a, K, K^*)$ *defined by Algorithm* 1 *is empty for every* $a, K, K^* > 0$.

Proof. If the roots of $p_0(z)$ are arranged so that

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|, \tag{2.7}$$

then the estimate

$$5^{-\mu_0} \leqslant \frac{|z_n|}{M[p_0(z), m_0]} \leqslant 1 \tag{2.8}$$

is valid (see [5]–[6]). For this reason the convergence of Algorithm 1 is identical with

$$|z_n^{(d)}| \leqslant cq^d \qquad (c > 0,\ 0 < q < 1), \tag{2.9}$$

where $z_n^{(d)}$ is the zero of $p_0(z + S^{(d)})$, $(d = 0, 1, \ldots)$ of minimal absolute value. Using the inequality (2.8), we have

$$\frac{n}{5c'}\left(\frac{1}{q}\right)^{d/\mu_0} \leqslant \frac{n}{5|z_n^{(d)}|^{1/\mu_0}} \leqslant |\sigma_{k(d)}^{(d)}| \qquad (d \geqslant d') \tag{2.10}$$

where $k(d) \in \{1, \ldots, n\}$ is the index of the maximal element in (1.3) and

$$\frac{n}{5c'}\left(\frac{1}{q}\right)^{d'/\mu_0} > 1. \tag{2.11}$$

Since $|\sigma_{k(d)}^{(d)}| = O(w^d)$, where $w = (1/q)^{1/\mu_0}$, therefore for a large index d_0

$$|\sigma_{k(d)}^{(d)}| > K \qquad (d \geqslant d_0) \tag{2.12}$$

is satisfied. Thus the theorem is proved.

THEOREM 2.2. *If* $K \geqslant K^* 2^{n+1}(1 + a^n 2^n)^{n+1} + 1$, *then*

$$\mathbf{P}_L(a, K, K^*) = \mathbf{P}(a, K^*) \tag{2.13}$$

is satisfied for Algorithm 2.

Proof. It is easy to see that the quantities recurring in the algorithm satisfy the inequalities

$$|p_0(\beta_j^{(d)})| \leqslant \begin{cases} \|p_0\| 2^{n+1} & (a < 0.5), \\ \|p_0\|(1 + 2^n a^n)^{n+1} & (a \geqslant 0.5), \end{cases} \tag{2.14}$$

$$\| T^j[p_0(z)] \| \leqslant \tfrac{1}{2}(2\|p_0(z)\|)^{2^j} \qquad (j = 1, \ldots, n) \tag{2.15}$$

and

$$\|p_0(\alpha_j^{(d)}z + \beta_j^{(d)})\| \leqslant \|p_0(z)\|(2 + 2^{n+1}a^n)^n \qquad (j = 0, 1, \ldots, 8), \tag{2.16}$$

for $d = 0, 1, \ldots$. With the notation

$$\delta = \|p_0(z)\|(2 + 2^{n+1}a^n)^n,$$

and by using (2.15)–(2.16), we have

$$\|T^k[p_0(\alpha_j^{(d)}z + \beta_j^{(d)})]\| \leqslant \tfrac{1}{2}(2\delta)^{2^k} \qquad (k = 1, \ldots, n). \tag{2.17}$$

Since K is greater than the right side of (2.14) and (2.16), using $\psi > 2\delta$ we can get $\widetilde{\delta} < 0.5$ which proves the theorem.

The difference between Algorithms 1 and 2 is caused by the fact that Algorithm 1 is based on the inequality (2.8) while Algorithm 2 is based on the characteristic function $N[p_0(z)]$ which is invariant for the mapping $p_0(z) \to p_0(z)/\psi$, $(\psi > 0)$.

We remark that Algorithm 1 modified by the mappings

$$p_0(z) \to p_0(z)/\psi, \qquad p_0(z) \to p_0(z/\psi) \qquad (0 < \psi \leqslant K)$$

also has a $\mathbf{P}_T(a, K, K^*)$ empty for every $a, K, K^* > 0$.

3. The Study of Cost Functions. In the previous section it was proved that Algorithm 1 is unapplicable. Since an approximate solution with a given error $\epsilon > 0$ can be computed in the bounded set $S[0, \widetilde{K}]$, where $\widetilde{K}$ depends on $p_0(z)$, ϵ, and the method M, further analysis of the algorithms is necessary.

The cost function of the jth algorithm ($j = 1, 2$) is defined by the number of additions and multiplications per step and denoted by K_a^j and K_m^j.

Assuming that the computing time of the kth root can be characterized by three additions and three multiplications (which is a rough underestimate), the cost function of Algorithm 1 is

$$K_m^1 = (l + 1)(m_0 + 4)\frac{n^2}{2} + (l + 1)(m_0 + 8)\frac{n}{4} + O(1), \tag{3.1}$$

$$K_a^2 = (l + 1)(m_0 + 4)\frac{n^2}{4} + (2l + 3)n + O(1). \tag{3.2}$$

For the cost function of Algorithm 2 the inequalities

$$K_m^2 \leqslant 27n^2 - 18n, \tag{3.3}$$

$$K_a^2 \leqslant 9n^2 + 36n, \tag{3.4}$$

hold.

If we identify the bounds (3.3)–(3.4) with the cost of one step, then the speed of Algorithm 2 is

$$|z^{(d)} - z^*| \leqslant c_2(2/5)^d \qquad (d = 0, 1, \ldots). \tag{3.5}$$

The speed of Algorithm 1 is

$$|S^{(d)} - z^*| \leqslant c_1[q(\alpha_{m_0}, m_0, l)]^d \qquad (d = 0, 1, \ldots), \tag{3.6}$$

where

$$(3.7)\quad q(\alpha_{m_0}, m_0, l) = \left[1 + 0.25(1 + \alpha_{m_0})^2 - (1 + \alpha_{m_0})\cos\frac{\pi}{l+1}\right]^{1/2} \alpha_{m_0}^{-1}.$$

If $\delta = (m_0 + 4)(l + 1)/54 > 1$ and $n \geqslant n'$, then

$$(3.8)\quad K_m^1 \geqslant \delta K_m^2 \quad \text{and} \quad K_a^1 > \delta K_a^2.$$

THEOREM 3.1. *If $l \geqslant l'$, then*

$$(3.9)\quad q(\alpha_{m_0}, m_0, l) > (2/5)^\delta.$$

Proof. For a large l'

$$(3.10)\quad q(\alpha_{m_0}, m_0, l)^2 \geqslant \frac{1 - (\cos \pi/(l+1))^2}{\alpha_{m_0}^2} > \frac{9\alpha_{m_0}^{-2}}{(l+1)^2} \qquad (l \geqslant l')$$

and

$$(3.11)\quad (5/2)^\delta > l + 1.$$

From this fact the theorem immediately follows.

If $l \geqslant l'$, then the cost of d steps of Algorithm 1 gives $[\delta d]$ steps using the Lehmer-Schur method. By Theorem 3.1 we have

$$(3.12)\quad c^*[q(\alpha_{m_0}, m_0, l)]^d > (2/5)^{[\delta d]} \qquad (c^* > 0, d \geqslant d_0),$$

which proves that *the Lehmer-Schur process is faster than the Turán process.* For the parameters $m_0 = 4$, $\alpha_4 = 0.9$, $l = 11$, (see [5]–[6]) the relation (3.12) is also satisfied. This can be verified easily by (3.10) and (3.11).

In the paper [6] there is a reference to the infinite precision integer arithmetics [1] for the sake of application of Algorithm 1. It is known [1] that the computing time of the multiplication is at most

$$(3.13)\quad l(x)^{1+\tau} \qquad (1 \geqslant \tau > 0)$$

units of time ($l(x)$ denotes the length of x in the binary system). Since Algorithm 1 has to use numbers of length at least $2^{m_0-2}l(x)$ where $l(x)$ is needed by Algorithm 2, for the cost functions in the measure of computing time,

$$(3.14)\quad K_m^1(t) \geqslant (\delta 2^{m_0-2})^{1+\tau} K_m^2(t)$$

is satisfied. As a simple corollary, in (3.12) we can write $\delta 2^{m_0-2}$ instead of δ. This fact increases the relative convergence speed of the Lehmer-Schur process.

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