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Roots of Two Transcendental Equations as Functions of a Continuous Real Parameter

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Abstract. The roots, λ and η, of the transcendental equations $j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda)$ and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta},$$

where $l = 1, 2, \ldots$ are considered as functions of the continuous real parameter α. The symbols j_l and y_l denote the spherical Bessel functions of the first and second kind. The two transcendental equations are invariant under the transformations $\lambda \to -\lambda$ and $\eta \to -\eta$, respectively. Therefore, only positive roots are discussed. All the λ-roots increase monotonically as α increases in the open interval (0, 1). For each order l, the smallest η-root decreases monotonically as α increases in (0, 1), tending towards $\sqrt{l(l+1)}$ as α approaches unity. For $\alpha \in (0, 1)$, all the other η-roots have a minimum value equal to $\sqrt{l(l+1)}/\alpha$.

In [1] roots of the transcendental equations,

$$j_l(\alpha\lambda)y_l(\lambda) = j_l(\lambda)y_l(\alpha\lambda) \tag{1}$$

and

$$[xj_l(x)]'_{x=\alpha\eta}[xy_l(x)]'_{x=\eta} = [xj_l(x)]'_{x=\eta}[xy_l(x)]'_{x=\alpha\eta}, \tag{2}$$

where j_l and y_l denote spherical Bessel functions of the first and second kind, are presented. Here we discuss the dependence of the roots λ_{ln} of Eq. (1) and η_{ln} of Eq. (2) on the continuous real parameter α whose domain is the open interval $(0, 1) = \{\alpha: 0 < \alpha < 1\}$. The subscript $n = 1, 2, \ldots$ orders the roots such that $\lambda_{ln+1} > \lambda_{ln}$ and $\eta_{ln+1} > \eta_{ln}$. Since

$$j_l(ze^{m\pi i}) = e^{ml\pi i}j_l(z), \qquad y_l(ze^{m\pi i}) = (-1)^m e^{ml\pi i}y_l(z)$$

$(l, m = 0, 1, 2, \ldots)$ [2, p. 439,10.1.34, 10.1.35], it follows that Eqs. (1) and (2) are invariant under the transformations $\lambda \longrightarrow -\lambda$ and $\eta \longrightarrow -\eta$, respectively.

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Therefore, only positive roots need be considered.

(3a) If Eq. (1) is written as $F(\alpha, \lambda) = 0$,

where

(3b) $$F(\alpha, \lambda) = j_l(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_l(\alpha\lambda),$$

then

(4) $$\frac{d\lambda}{d\alpha} = -\frac{\partial F/\partial\alpha}{\partial F/\partial\lambda},$$

where

(5a) $$\frac{\partial F}{\partial\alpha} = \lambda[j_{l-1}(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_{l-1}(\alpha\lambda)]$$

and

(5b) $$\frac{\partial F}{\partial\lambda} = \alpha[j_{l-1}(\alpha\lambda)y_l(\lambda) - j_l(\lambda)y_{l-1}(\alpha\lambda)] - [j_{l-1}(\lambda)y_l(\alpha\lambda) - j_l(\alpha\lambda)y_{l-1}(\lambda)].$$

The expressions (5a) and (5b) for the partial derivatives $\partial F/\partial\alpha$ and $\partial F/\partial\lambda$ have been obtained by means of the formula [2, p. 439, 10.1.21]

(6) $$\frac{l+1}{z}f_l(z) + \frac{d}{dz}f_l(z) = f_{l-1}(z), \qquad f_l(z) = \begin{cases} j_l(z), \\ y_l(z), \end{cases}$$

and by utilizing Eqs. (3). By virtue of the relation [2, p. 439, 10.1.31]

(7) $$j_l(z)y_{l-1}(z) - j_{l-1}(z)y_l(z) = z^{-2}$$

and Eqs. (3) one obtains from Eqs. (4) and (5)

(8a) $$\frac{d\lambda}{d\alpha} = -\frac{\lambda/\alpha}{1 - \alpha\tau_l^2(\alpha, \lambda)},$$

where

(8b) $$\tau_l(\alpha, \lambda) = j_l(\alpha\lambda)/j_l(\lambda) = y_l(\alpha\lambda)/y_l(\lambda).$$

For $0 < \alpha < 1$ expression (5b) is finite and, if Eqs. (3) hold, expression (5a) cannot vanish. Therefore,

(9) $$\frac{d\lambda}{d\alpha} \neq 0 \quad \text{for } 0 < \alpha < 1,$$

which means that λ is a monotonic function of α. This implies that if, for given values of l and n,

(10) $$\lambda_{ln}(\alpha_2) > \lambda_{ln}(\alpha_1) \quad \text{and} \quad \alpha_2 > \alpha_1,$$

where $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$, then $\lambda_{ln}(\alpha)$ is a monotonically increasing function for $0 < \alpha < 1$. In particular, the roots λ_{ln} given in [1] for $l = 1(1)15$ and $n = 1(1)30$ satisfy condition (10).

From Eq. (5a) it follows that $\lim_{\alpha\to 1} \partial F/\partial\alpha \neq 0$, and from Eq. (5b) that $\lim_{\alpha\to 1} \partial F/\partial\lambda = 0$. Therefore, Eq. (4) entails that

$$\lim_{\alpha\to 1} \frac{d\lambda}{d\alpha} = \pm\infty. \tag{11}$$

Condition (10) excludes the minus sign in Eq. (11).

If Eq. (2) is written as

$$G(\alpha, \eta) = 0, \tag{12a}$$

where

$$G(\alpha, \eta) = s_l(\alpha\eta)t_l(\eta) - s_l(\eta)t_l(\alpha\eta), \tag{12b}$$

$$s_l(x) = xj_{l-1}(x) - lj_l(x), \qquad t_l(x) = xy_{l-1}(x) - ly_l(x), \tag{12c}$$

then

$$\frac{d\eta}{d\alpha} = -\frac{\partial G/\partial\alpha}{\partial G/\partial\eta}, \tag{13}$$

where

$$\frac{\partial G}{\partial\alpha} = \frac{1}{\alpha}[l(l+1) - (\alpha\eta)^2]\,[j_l(\alpha\eta)t_l(\eta) - y_l(\alpha\eta)s_l(\eta)] \tag{14a}$$

and

$$\begin{aligned}\frac{\partial G}{\partial\eta} = {} & \frac{1}{\eta}[l(l+1) - (\alpha\eta)^2]\,[j_l(\alpha\eta)t_l(\eta) - y_l(\alpha\eta)s_l(\eta)] \\ & -\frac{1}{\eta}[l(l+1) - \eta^2]\,[j_l(\eta)t_l(\alpha\eta) - y_l(\eta)s_l(\alpha\eta)].\end{aligned} \tag{14b}$$

The expression (12b) has been derived from Eq. (2) by means of Eq. (6), and the expressions (14a) and (14b) for the partial derivatives $\partial G/\partial\alpha$ and $\partial G/\partial\eta$ have been obtained by means of the formula [2, p. 439, 10.1.22]

$$\frac{l}{z}f_l(z) - \frac{d}{dz}f_l(z) = f_{l+1}(z), \qquad f_l(z) = \begin{cases} j_l(z), \\ y_l(z). \end{cases}$$

By virtue of Eqs. (7) and (12) the expressions (14a) and (14b) can be rewritten as

$$\frac{\partial G}{\partial\alpha} = \frac{1}{\alpha^2\eta}[l(l+1) - (\alpha\eta)^2]\,\rho_l^{-1}(\alpha, \eta), \tag{15a}$$

$$\frac{\partial G}{\partial\eta} = \frac{1}{\alpha\eta^2}[l(l+1) - (\alpha\eta)^2]\,\rho_l^{-1}(\alpha, \eta) - \frac{1}{\eta^2}[l(l+1) - \eta^2]\,\rho_l(\alpha, \eta), \tag{15b}$$

where

(15c) $$\rho_l(\alpha, \eta) = \frac{s_l(\alpha\eta)}{s_l(\eta)} = \frac{t_l(\alpha\eta)}{t_l(\eta)}.$$

By substituting (15a) and (15b) in Eq. (13), one obtains

(16a) $$\frac{d\eta}{d\alpha} = -\frac{\eta/\alpha}{1 - \alpha\sigma_l(\alpha, \eta)\rho_l^2(\alpha, \eta)},$$

where

(16b) $$\sigma_l(\alpha, \eta) = [l(l+1) - \eta^2]/[l(l+1) - (\alpha\eta)^2].$$

The expression (16a) for the total derivative is analogous to the expression (8a). The definitions (12c) and the relations (15c) imply that $\rho_l(\alpha, \eta)$ and $\rho_l^{-1}(\alpha, \eta)$ are both nonzero. Since $\eta \neq 0$, it follows from Eq. (13) and Eqs. (15a) and (15b) that

(17) $$\frac{d\eta}{d\alpha} = 0, \quad 0 < \alpha < 1,$$

if and only if

(18) $$\alpha\eta = \sqrt{l(l+1)}.$$

From Eqs. (17) and (18) it is obvious that η is not a monotonic function of α, unless

(19) $$\lim_{\alpha\to 1} \eta = \sqrt{l(l+1)},$$

in which case one finds by means of l'Hospital's rule that

(20) $$\lim_{\alpha\to 1} \frac{d\eta}{d\alpha} = -\sqrt{l(l+1)}.$$

The roots which satisfy Eq. (19) are, therefore, monotonically decreasing functions of α. All other roots have an extremum in the open interval (0, 1), in accordance with Eq. (17), and satisfy

(21) $$\lim_{\alpha\to 1} \frac{d\eta}{d\alpha} = \pm\infty,$$

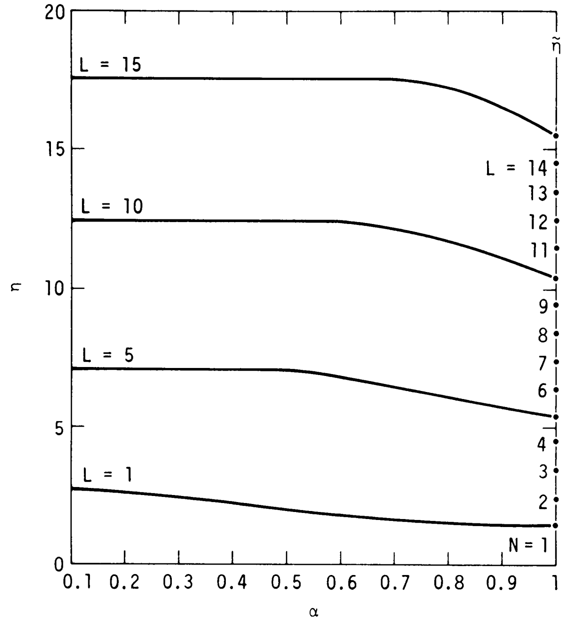
since

$$\lim_{\alpha\to 1} \frac{\partial G}{\partial \alpha} \neq 0, \quad \text{if } \lim_{\alpha\to 1} \eta \neq \sqrt{l(l+1)},$$

and

$$\lim_{\alpha\to 1} \frac{\partial G}{\partial \eta} = 0.$$

The minima of the η-roots, together with the corresponding values of the parameter α, are presented for $l = 1(1)15$ and $n = 1(1)30$ in the microfiche supplement. In the graphs, the solid curves represent η-roots as functions of α for $l = 1, 5, 10, 15$ and for $n = 1, 2, 3, 4, 5, 10$. The dashed curves connect the minima, $\tilde{\eta}$, of these roots.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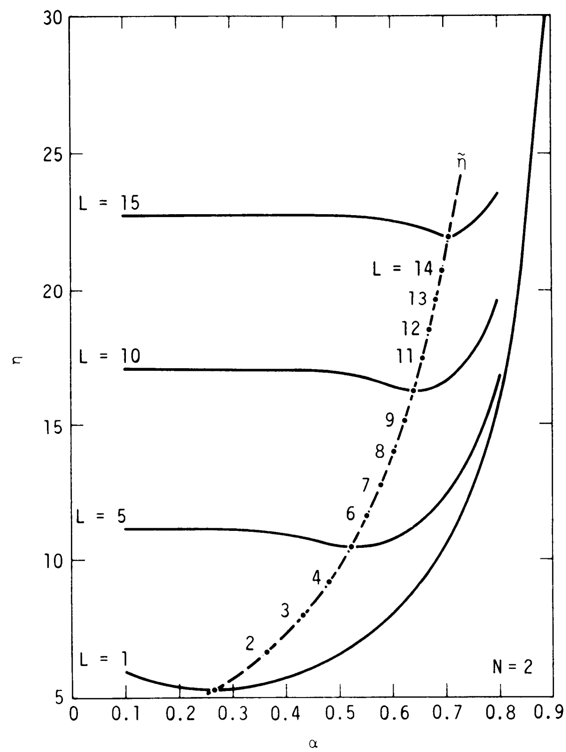
α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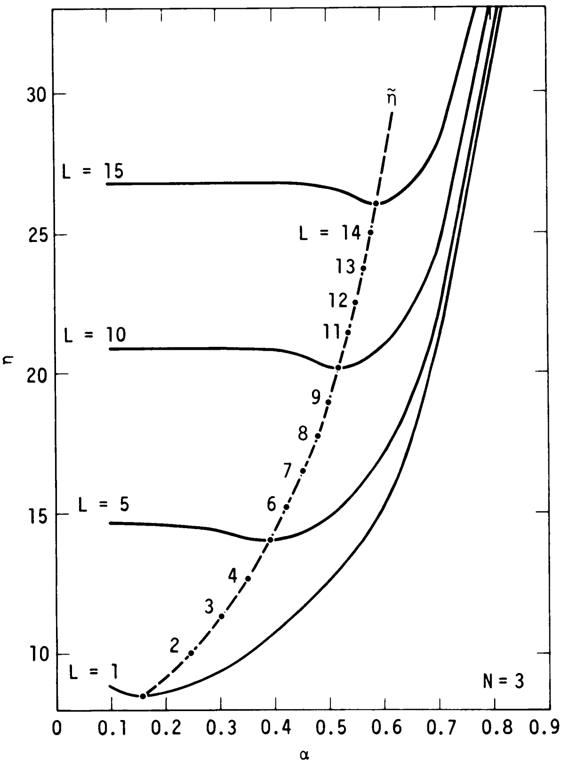

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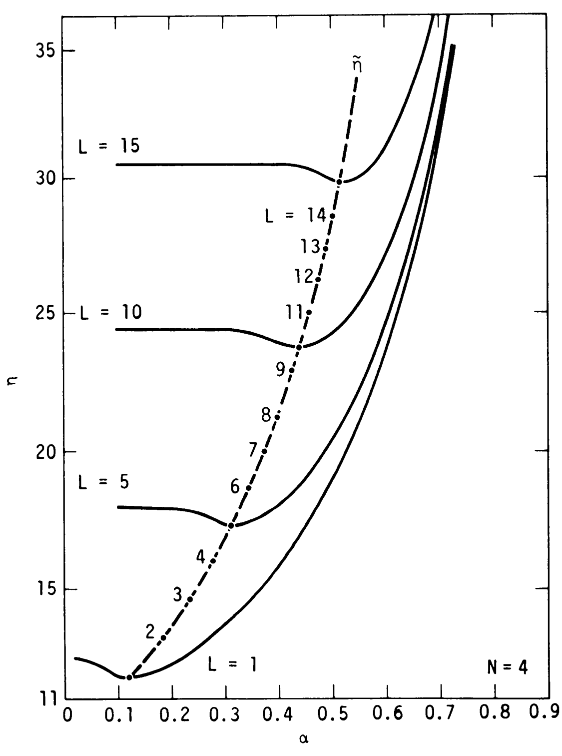

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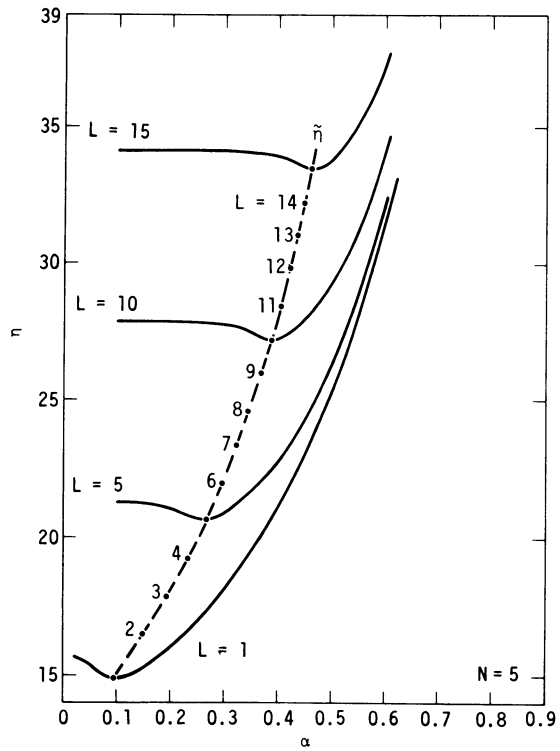
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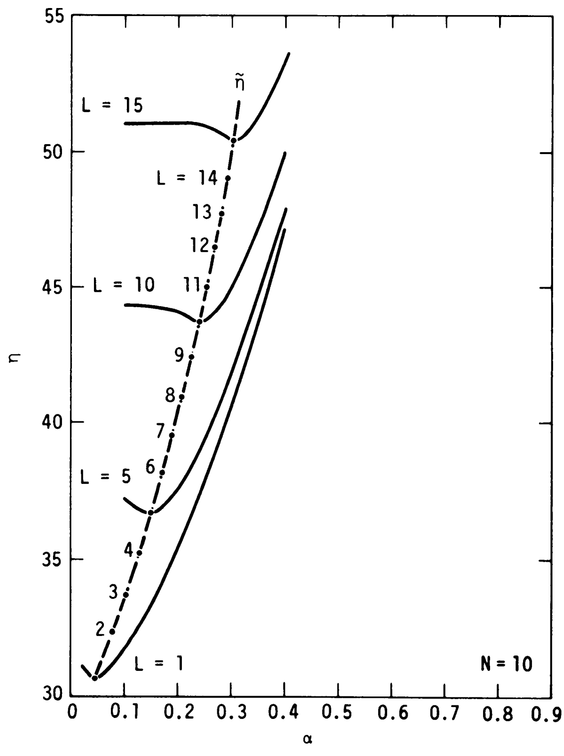
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