MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 142
APRIL 1978, PAGES 635–636

Computer-Generated van der Waerden Partitions

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Abstract. An exhaustive search made on a PDP-8 mini-computer shows the value of the van der Waerden number $n(2, 5)$ to be 178.

Let S_n denote the set of natural numbers less than or equal to n. Van der Waerden's Theorem [2] may be stated as follows: For given natural numbers k and l, there exists a natural number $n = n(k, l)$ such that for any $f\colon S_n \longrightarrow S_k$, there is an arithmetic progression of length l of arguments in S_n on which f is constant. In this note we compute the least positive $n = n(k, l)$ for some values of k and l.

The proof given in [2] has much overkill and gives an upper bound for $n(k, l)$ that is theoretically constructive but for all practical purposes only existential.

For given i, k, and l, we define $f\colon S_i \longrightarrow S_k$ to be *valid* if there is no arithmetic progression of the elements in S_i of length l on which f is constant. Consider the following algorithm (for fixed k and l) which is designed to find the largest value of i for which $f\colon S_i \longrightarrow S_k$ is valid. (Thus after termination, $i\max + 1$ gives the smallest value of $n(k, l)$):

STEP 1. Set $i = 1, f(1) = 1, i\max = 1$.
STEP 2. Increment i by 1 and then set $f(i) = 1$.
STEP 3. If $f\colon S_i \longrightarrow S_k$ is valid, then go to STEP 2.
STEP 4. If $f(i) = k$, then go to STEP 6.
STEP 5. Increment $f(i)$ by 1 and then go to STEP 3.
STEP 6. If $i - 1 > i\max$, then set $i\max = i - 1$.
STEP 7. Decrement i by 1.
STEP 8. If $i > 1$, then go to STEP 4.
STEP 9. Stop.

This algorithm was programmed on a PDP-8 mini-computer and run with $k = 2$ and various values of l. The resulting values of $i\max$ at termination and approximate running times are shown in the following table:

l	3	4	5
$i\max$	8	34	177
time	instantaneous	3 seconds	210 hours

What is perhaps somewhat more interesting is the nature of the valid functions $f\colon S_{i\max} \longrightarrow S_2$. For purposes of illustration we use the two symbols 0 and 1 (rather than 1 and 2) and indicate the functions merely by strings of 0's and 1's. We

Received April 19, 1976.
AMS (MOS) subject classifications (1970). Primary 65D99; Secondary 68A10.

indicate only the essentially different strings that were found. All others can be obtained by relabeling (i.e., by interchanging the 0's and 1's).

In the case $l = 3$, there are three strings: 00110011, 01011010, and 01100110.

In the case $l = 4$, there is a repeating subsequence of length 10, which we denote by $X = 0100011101$. Then all strings are of the form $*X*X*X*$, where the *'s (called freebies) denote single entries whose only restriction is that there be no arithmetic string of four freebies all alike. There are 14 valid strings of length 34.

In the case $l = 5$, the repeating subsequence structure is "two deep." More precisely, let $X = 0010000100$ and $Y = 0111010001$. Let X' and Y' be obtained from X and Y, respectively, by relabeling. Then the subsequence Z is $X*Y*X'*Y'$ (as before, the *'s denote freebies). The valid strings of length 177 are all of the form $*Z*Z*Z*Z*$ or reversed from this (end for end). Note that by virtue of the symmetry exhibited by Z, we may cyclically "conjugate" and obtain, for example, a reversal by instead setting $Z = Y*X'*Y'*X$. These conclusions about the subsequence structure were arrived at in the following way:

The initial string 00010000100 was given (i.e., first freebie set to 0 and next 10 symbols set to X) to the computer. Sufficient output was printed to establish that Z must be constructed and repeated as given above in order to obtain a valid string of length 177. Then, noting that there are 17 freebies, a count was made of all valid strings of length 17 with first symbol 0, yielding 24,203. This compares exactly with a count made of all valid strings of length 177 with initial string 00010000100.

A check via computer output also showed that the first 11 symbols must be initialized to $*U$ (U is one of the four "blocks" X, Y, X', or Y' shown above), and that the appropriate "conjugate" of Z must be constructed and repeated in order to obtain valid strings of length 177.

In conclusion, we note that $n(2, 5) = 178$ found above extends the table on page 31 prepared by V. Chvatal [1]. We would also like to record our agreement with the value $n(3, 3) = 27$ found in this reference.

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1. V. CHVATAL, *Some Unknown van der Waerden Numbers*, Symposium in Combinatorial Methods, Calgary, Alberta, Canada, 1969, pp. 31–33.

2. A. Y. KHINCHIN, *Three Pearls of Number Theory*, Graylock Press, Rochester, New York, 1952.