

On the Convergence of Difference Approximations to Nonlinear Contraction Semigroups in Hilbert Spaces

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Abstract. Convergence properties of the difference schemes

$$(S) \quad h^{-1} \sum_{j=0}^k \alpha_j u_{n+j} + \sum_{j=0}^k \beta_j A u_{n+j} = 0, \quad n \geq 0,$$

for evolution equations

$$(E) \quad \frac{du(t)}{dt} + Au(t) = 0, \quad t \geq 0; \quad u(0) = u_0 \in \overline{D(A)}$$

are studied. Here A is a nonlinear, maximally monotone operator in a real Hilbert space. It is shown, in particular, that if the scheme (S) is consistent and stable for the test equation $x' = \lambda x$ for $\lambda \in \mathbb{C} - K$, where K is a compact subset of the right half-plane, then (S) is convergent as $h \downarrow 0$, with suitable initial values, for (E), on compact intervals $[0, T]$. Moreover, the convergence is uniform on the half-axis $t \geq 0$, if the solution $u(t)$ tends strongly to a constant as $t \rightarrow \infty$. We also show that under weaker stability conditions one can construct conditionally convergent methods.

Introduction. We study the approximation of nonlinear contraction semigroups by means of linear multistep methods. In an earlier paper [12] we derived bounds for the accumulated errors in terms of the local (truncation) errors. If the solutions are continuously differentiable, then the local errors are small enough for the error bounds to imply the convergence as the step length tends to zero, uniformly on compact time intervals.

Here we study the convergence without assuming smoothness of the solutions. In order to show the convergence on compact time intervals we replace the operator by its Yosida approximation. Since the Yosida approximation is Lipschitz-continuous, the solutions are in C^1 and our difference methods yield convergent approximations. The main task then is to bound the difference between the solutions of the original and approximated difference equations.

For one special method, namely the implicit Euler method, the convergence on compact time intervals is well known, stating that the semigroup $S(t)$ is obtained from the generator $-A$ by the product formula

$$S(t)x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A \right)^{-n} x \quad \text{for all } x \in \overline{D(A)},$$

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see [1], [2]. Difficulties in generalizing this result to other methods are hidden in the fact that the contractiveness is generally not preserved under discretization. In Hilbert spaces we still have enough contraction-like behavior to proceed (see Theorem 1 below), but in Banach spaces the only results we know are for the implicit Euler method, see, e.g. [4], [9], [13].

Under some conditions $S(t)x$ tends strongly to a limit as $t \rightarrow \infty$, see [3]. We show that our approximations then converge not only on compact time intervals but uniformly on the half-axis $t \geq 0$.

We also demonstrate that our class of methods cannot substantially be enlarged, and give a convergence rate for problems with initial values inside the domain of the operator A .

We finally show that all stable multistep methods which are not weakly stable can be used to build conditionally convergent methods, where one approximates the nonlinear operator by its Yosida approximation and relates the growth of the Lipschitz constant to the decrease of the step length.

2. Results. Let H be a real Hilbert space and A a generally nonlinear single-valued operator $D(A) \subset H \rightarrow H$. The operator A is said to be monotone if

$$(1) \quad (u - v, Au - Av) \geq 0 \quad \text{for all } u, v \in D(A),$$

and maximally monotone if additionally $R(I + A) = H$. By the solution of the initial value problem

$$(2) \quad \frac{du(t)}{dt} + Au(t) = 0, \quad t \geq 0, \quad u(0) = u_0 \in \overline{D(A)}$$

we shall mean the unique continuous function $u: t \rightarrow S(t)u_0$, where $S(t)$ denotes the contraction semigroup generated by $-A$, see [1]. The initial value problem is approximated using linear multistep methods (ρ, σ) :

$$(3) \quad h^{-1}\rho(E)u_n^h + \sigma(E)Au_n^h = 0, \quad n \geq 0, \quad u_0^h, \dots, u_{k-1}^h \in D(A) \text{ given.}$$

In (3) E denotes the translation operator $Ey_n = y_{n+1}$, $h > 0$ is the step length and ρ and σ are polynomials

$$(4) \quad \rho(\xi) = \sum_{j=0}^k \alpha_j \xi^j, \quad \sigma(\xi) = \sum_{j=0}^k \beta_j \xi^j$$

with real coefficients and with no common divisor.

We first give a result concerning the solutions of (3).

Consider

$$(5) \quad h^{-1}\rho(E)u_n + \sigma(E)Au_n = r_{n+k}, \quad n \geq 0,$$

and

$$(6) \quad h^{-1} \rho(E)(u_n - v_n) + \sigma(E)(Au_n - Av_n) = f_{n+k}, \quad n \geq 0.$$

THEOREM 1. Suppose that the method (ρ, σ) satisfies

$$(7) \quad \rho(1) = 0, \quad \rho'(1) = \sigma(1),$$

$$(8) \quad \operatorname{Re} \frac{\rho}{\sigma}(\xi) > 0 \quad \text{for } |\xi| > 1,$$

$$(9) \quad \sigma(\xi) \neq 0 \quad \text{on } |\xi| = 1.$$

If A is maximally monotone, then for any given $\{r_n\} \subset H$ and $\{u_0, \dots, u_{k-1}\} \subset D(A)$ there exists a unique $\{u_n\} \subset D(A)$ satisfying (5). Moreover, there exists a constant C , depending only on the method (ρ, σ) such that for $\{u_n - v_n\}$ satisfying (6) we have, for all $n \geq k$,

$$(10) \quad |u_n - v_n| \leq C \left\{ \max_{0 \leq j < k} [|u_j - v_j| + h|Au_j - Av_j|] + h \sum_{j=k}^n |f_j| \right\}. \quad \square$$

This result is included in Theorems 2.1 and 3.1 of [12]. We first observe that if we can show convergence for some special initial values u_0^h, \dots, u_{k-1}^h in the absence of round-off errors ($r_n = 0$), then the general case can be obtained from (10).

THEOREM 2. Assume that (7), (8) and (9) hold. Let A be maximally monotone and u the solution of (2). Set $u_j^h = (I + hA)^{-1}u_0$ for $j = 0, \dots, k - 1$, and let $\{u_n^h\}$ satisfy (3). Then, for all $T < \infty$,

$$(11) \quad \sup_{0 \leq n < Th^{-1}} |u(nh) - u_n^h| \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad \square$$

The proof of Theorem 2 will give us the following additional result.

COROLLARY 1. If, in addition to the assumptions of Theorem 2, $u_0 \in D(A)$, then for all $T < \infty, h < \infty$

$$(12) \quad \sup_{0 \leq n < T/h} |u(nh) - u_n^h| \leq C|Au_0| \{h + T^{1/2}h^{1/2} + [T + T^{1/2}]h^{1/3}\},$$

with C depending only on the method (ρ, σ) . \square

For the implicit Euler method ("Rothe method"), applied to a class of nonlinear initial boundary value problems Kačur [8] has obtained a convergence order $O(h^{1/2})$. Recall that if the solutions are in C^1 , then Theorem 1 implies the convergence order $O(h)$.

The next example shows that the assumption $u_0 \in D(A)$ in Corollary 1 is essential for getting the convergence order $O(h^{1/3})$. Let $H = \mathbf{R}$ and put $Au = -u^{1-s}$ for $u > 0$ and $s > 1$. Then $u(t) = s^{1/s}t^{1/s}$ is the solution of $u' + Au = 0$ with $u(0) = 0$. But the implicit Euler method,

$$h^{-1}(u_{n+1} - u_n) - (u_{n+1})^{1-s} = 0,$$

with the exact initial value $u_0 = 0$ yields $u_1 = h^{1/s}$. Hence $|u(h) - u_1| = (s^{1/s} - 1)h^{1/s}$ for the exact initial value, and for the one of Theorem 2, $u_0^h = (I + hA)^{-1}0 = h^{1/s}$, we have $|u(0) - u_0^h| = h^{1/s}$.

THEOREM 3. *If in addition to the assumptions of Theorem 2*

$$(13) \quad u(t) \rightarrow c \quad \text{as } t \rightarrow \infty,$$

then

$$(14) \quad \sup_{n \geq 0} |u(nh) - u_n^h| \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad \square$$

It is shown in [3] that if $A = \partial\varphi$ and $\varphi(-u) = \varphi(u)$ for all u then (13) holds. Another sufficient condition is that $0 \in R(\partial\varphi)$ and the set $\{u \in H \mid |u| \leq M, \varphi(u) \leq M\}$ is compact for all $M < \infty$ [1, p. 90]. Finally, note that if $A - \alpha I$ is monotone for some $\alpha > 0$, then $u(t)$ tends exponentially to the unique solution of $0 = Ac$.

Consider the assumptions on (ρ, σ) . The (algebraic) consistency conditions (7) are certainly necessary for the convergence (11). We shall now demonstrate that also the assumption (8) is necessary. Methods satisfying (8) are called A -stable [5]. If (8) does not hold, then there exists a $q \in \mathbb{C}$ with $\operatorname{Re} q < 0$ such that the equation

$$\rho(\zeta) - q\sigma(\zeta) = 0$$

has at least one solution $\zeta = \xi$ satisfying $|\xi| > 1$. Hence, the difference equation

$$\rho(E)\eta_n = \lambda h\sigma(E)\eta_n$$

has, for $\lambda h = q$ and for initial values $\eta_j = \xi^j x$, $j = 0, \dots, k-1$, the solution $\eta_n = \xi^n x$. Let l^2 be the (complex) Hilbert space of sequences $x = (x^\mu)_{\mu \geq 0}$, with $x^\mu \in \mathbb{C}$ and $(x, y) = \sum_{\mu \geq 0} \bar{y}^\mu x^\mu$, and let Λ be the operator mapping (x^μ) to $(\lambda_\mu x^\mu)$. Then the solutions of

$$h^{-1}\rho(E)x_n = \sigma(E)\Lambda x_n$$

satisfy

$$\rho(E)x_n^\mu = h\lambda_\mu\sigma(E)x_n^\mu.$$

Put $x_0 = (\mu^{-\alpha})$ with $\alpha > 2$ and choose $\lambda_\mu = q\mu$. Then $\Lambda x_0 \in l^2$ and hence $x_0 \in D(\Lambda)$. Now choose a sequence h_μ tending to zero by setting $h_\mu = \mu^{-1}$. Then, for all n

$$\|x_n\|_2 \geq \|x_n\|_\infty \geq |\xi|^n |x_0^\mu|.$$

But, as $h_\mu \rightarrow 0$, $|\xi|^{Th_\mu^{-1}} |x_0^\mu| = (|\xi|^T)^\mu \mu^{-\alpha} \rightarrow \infty$ and therefore,

$$\|x_{Th_\mu^{-1}} - x(T)\|_2 \rightarrow \infty \quad \text{as } h_\mu \downarrow 0.$$

The assumption (9), on the contrary, seems to be a technical one. A practical method excluded by (9) is the trapezoidal rule

$$(15) \quad h^{-1}(u_{n+1}^h - u_n^h) + \frac{1}{2}(Au_{n+1}^h + Au_n^h) = 0, \quad n \geq 0.$$

THEOREM 4. *Let A be maximally monotone and u the solution of (2). Set $u_0^h = (I + hA/2)^{-1}u_0$, then for all $h > 0$ a unique $\{u_n^h\} \subset D(A)$ exists satisfying (15), and for all $T < \infty$ (11) holds. Moreover, we have (14) whenever (13) holds. \square*

It is natural to ask whether we could relax (9) by approximating the unbounded operator A by a Lipschitz-continuous operator and obtain conditionally convergent methods. In order to state a result of this type we introduce some notation. If A is maximally monotone, then the operator

$$J_\lambda = (I + \lambda A)^{-1}, \quad \lambda > 0,$$

is a contraction defined for all $x \in H$; and therefore, the operator

$$A_\lambda = \lambda^{-1}(I - J_\lambda)$$

referred to as the Yosida approximation of A , is Lipschitz-continuous. We shall also need the concept of stability region of a method (ρ, σ) which is defined to be the set S of complex numbers q for which the characteristic polynomial $\rho(\zeta) - q\sigma(\zeta)$ satisfies the root condition, i.e. its roots satisfy $|\zeta_\nu| \leq 1$ and those of modulus one are simple. Our stability assumption on the method (ρ, σ) is

$$(16) \quad \text{there exists } r > 0 \text{ such that } \{z \in \mathbb{C} \mid |z + r| \leq r\} \subset S.$$

It can be shown that (16) holds for all stable methods (ρ, σ) which are not “weakly stable” (defined in terms of the growth parameters; G. Dahlquist, personal communication).

We shall approximate the solution of (2) by the scheme

$$(17) \quad h^{-1}\rho(E)u_{\lambda,n}^h + \sigma(E)A_\lambda u_{\lambda,n}^h = 0, \quad n \geq 0.$$

Note that since A_λ is Lipschitz-continuous with the constant $1/\lambda$, (17) has a unique solution $\{u_{\lambda,n}^h\} \subset H$, whenever h/λ is small enough, for all given $\{u_{\lambda,0}^h, \dots, u_{\lambda,k-1}^h\} \subset H$.

THEOREM 5. *Suppose that the method (ρ, σ) satisfies (7) and (16). Let A be maximally monotone and u the solution of (2), with $u_0 \in D(A)$. Set $u_{\lambda,j}^h = u_0$ for $j = 0, \dots, k - 1$, and suppose $\{u_{\lambda,n}^h\}$ satisfies (17). Assume that for a given $c > 0$, we have $\lambda^2 \geq ch$. Then there exist constants $\lambda_0 = \lambda_0(\rho, \sigma, c) > 0$ and $C = C(\rho, \sigma, T, c)$ such that if $\lambda \in (0, \lambda_0)$, then*

$$(18) \quad \sup_{0 \leq n < T/h} |u(nh) - u_{\lambda,n}^h| \leq C|Au_0| \{(h/\lambda) + \lambda^{1/2}\}. \quad \square$$

We finally point out that (16) cannot be replaced by just assuming that ρ is stable, that is $0 \in S$. Consider the “leap-frog” method $(\rho, \sigma) = (\zeta^2 - 1, 2\zeta)$ for which

S is the open interval $(-i, i)$. This method can be used as a basis for conditionally stable schemes for some linear hyperbolic equations. Let H be the real Hilbert space l^2 of square summable sequences $(x^\mu)_{\mu \geq 0}$ and let A be the linear operator $(x^\mu) \rightarrow (\alpha^\mu x^\mu)$ with $\alpha > 1$. Hence A_λ is the operator $(x^\mu) \rightarrow (\alpha^\mu(1 + \lambda\alpha^\mu)^{-1}x^\mu)$. It is easily seen that there exist initial values x_0, x_1 , depending on λ and h but uniformly bounded, such that $\|x_{T/h}\|_2 \rightarrow \infty$ as $\lambda, h \rightarrow 0$.

3. Proof of Theorem 2. We first show that if the Theorem 2 is true for all $u_0 \in D(A)$, the general case will follow. Assume, therefore, that $u_0 \in D(A)$ and fix $\epsilon > 0$. Pick a $v_0 \in D(A)$ so that $|u_0 - v_0| < \epsilon$. Then due to the contraction property of the solutions, we have

$$(19) \quad |u(t) - v(t)| \leq |u_0 - v_0| < \epsilon \quad \text{for } t \geq 0.$$

For $|u_n^h - v_n^h|$ we get from Theorem 1

$$(20) \quad |u_n^h - v_n^h| \leq C\{|u_0^h - v_0^h| + h|Au_0^h - Av_0^h|\} \quad \text{for } n \geq 0, h > 0.$$

(C will denote a constant which is independent on n, h, λ, T but not necessarily the same at each occurrence.)

We recall some well-known properties of the resolvent $J_\lambda = (I + \lambda A)^{-1}$ and the Yosida approximation $A_\lambda = (1/\lambda)(I - J_\lambda)$, of a maximally monotone operator A :

- (i) $|A_\lambda x| \uparrow |Ax|$ and $A_\lambda x \rightarrow Ax$ for $x \in D(A)$ as $\lambda \downarrow 0$,
- (ii) $|A_\lambda x| \uparrow \infty$ for $x \notin D(A)$ as $\lambda \downarrow 0$,
- (iii) $A_\lambda x = AJ_\lambda x$ for all $x \in H, \lambda > 0$.

Since J_λ is a contraction for $\lambda > 0$, A_λ is (maximally monotone, and) Lipschitz-continuous with constant $1/\lambda$.

Using (iii), we get

$$h|Au_0^h - Av_0^h| = h|AJ_h u_0 - AJ_h v_0| = h|A_h u_0 - A_h v_0| \leq |u_0 - v_0| < \epsilon.$$

Substituting this into (20) yields

$$(21) \quad |u_n^h - v_n^h| \leq 2C\epsilon \quad \text{for all } n \geq 0, h > 0.$$

Suppose now that Theorem 2 holds for all $v_0 \in D(A)$ so that

$$\sup_{0 \leq n < T/h} |v(nh) - v_n^h| \rightarrow 0 \quad \text{as } h \downarrow 0.$$

Fix h_0 so that

$$|v(nh) - v_n^h| < \epsilon \quad \text{for } 0 < h < h_0, 0 \leq n < T/h.$$

Then, for $0 < h < h_0$,

$$\begin{aligned} & \sup_{0 \leq n < T/h} |u(nh) - u_n^h| \\ & \leq |u(nh) - v(nh)| + |v(nh) - v_n^h| + |v_n^h - u_n^h| \leq \epsilon + \epsilon + 2C\epsilon, \end{aligned}$$

which completes the reduction.

From now on we assume that $u_0 \in D(A)$. We intend to use the known convergence for Lipschitz-continuous problems. The latter is well known; it also follows easily from (10), with $u_n = u_n^h$, $v_n = u(nh)$, and with $\{q_n\} = \{\tau_n^h\}$ representing the local discretization errors which are sufficiently small because solutions to Lipschitz-continuous problems are continuously differentiable. In fact, replace A by A_λ and denote the solutions of the differential and the difference equation by $u_\lambda(t)$ and $u_{\lambda,n}^h$, respectively. Then $\{\tau_{\lambda,n}^h\}$ is defined by

$$(22) \quad h^{-1} \rho(E)u_\lambda(nh) + \sigma(E)A_\lambda u_\lambda(nh) = \tau_{\lambda,n}^h \quad \text{for } n \geq 0, h, \lambda > 0.$$

Using the mean-value theorem and the fact that $|A_\lambda u_\lambda(t)| \leq |A_\lambda u_\lambda(0)|$ (see Theorem 3.1 in [1]), we have

$$(23) \quad |\tau_{\lambda,n}^h| \leq C(h/\lambda)|A_\lambda u_\lambda(0)|,$$

with C depending only on (ρ, σ) . We shall put $u_\lambda(0) = u_0$ and $u_{\lambda,j}^h = u_0^h$ for $j = 0, \dots, k - 1$. Then (10) gives us

$$(24) \quad |u_\lambda(nh) - u_{\lambda,n}^h| \leq C \left\{ |u_0 - u_0^h| + h|A_\lambda u_0 - A_\lambda u_0^h| + h \sum_{j=k}^n |\tau_{\lambda,j}^h| \right\}.$$

But $|u_0 - u_0^h| = |u_0 - J_h u_0| = h|A u_0|$ and $h|A_\lambda u_0 - A_\lambda u_0^h| \leq 2h|A u_0|$, so that (23) and (24) yield

$$(25) \quad |u_\lambda(nh) - u_{\lambda,n}^h| \leq C\{1 + (T/\lambda)\}|A u_0|h, \quad 0 \leq n < T/h.$$

For $|u(t) - u_\lambda(t)|$ the following bound is given in [1, p. 56]:

$$(26) \quad |u(t) - u_\lambda(t)| \leq (1/\sqrt{2})|A u_0|\sqrt{(\lambda t)}, \quad t \geq 0.$$

In order to obtain the convergence of u_n^h we also need a bound for $|u_n^h - u_{\lambda,n}^h|$. We start with the equation

$$(27) \quad h^{-1} \rho(E)(u_n^h - u_{\lambda,n}^h) + \sigma(E)(A u_n^h - A_\lambda u_{\lambda,n}^h) = 0, \quad n \geq 0.$$

Define $u_{\lambda,n}^h = u_n^h = u_0^h$ for $n < 0$, and denote $-A u_n^h + A_\lambda u_{\lambda,n}^h = z_n$. Then we have for all $n \in \mathbf{Z}$

$$(28) \quad h^{-1} \rho(E)(u_n^h - u_{\lambda,n}^h) = p_n,$$

with $p_n = 0$ for $n < 0$ and $p_n = \sigma(E)z_n$ for $n \geq 0$.

By assumption (9), $\sigma(\zeta)^{-1}\rho(\zeta)$ is analytic in some neighborhood of the unit circle and has an expansion $\sigma(\zeta)^{-1}\rho(\zeta) \sim \gamma_0 + \gamma_1\zeta^{-1} + \gamma_2\zeta^{-2} + \dots$ with $\gamma = \{\gamma_n\} \in l^1$. Hence, (28) implies

$$(29) \quad h^{-1} \sum_{j=0}^n \gamma_{n-j}(u_j^h - u_{\lambda,j}^h) = w_n \quad \text{for } n \geq 0,$$

where w_n is obtained by solving

$$\begin{aligned} \sigma(E)w_n &= \sigma(E)z_n & \text{for } n \geq 0, \\ w_n &= 0 & \text{for } n < 0, \\ \sigma(E)w_n &= 0 & \text{for } n < 0. \end{aligned}$$

Hence, $w_n = z_n + r_n$, where

$$z_n = -Au_n^h + A_\lambda u_{\lambda,n}^h$$

and

$$|r_n| \leq Cr^n \max_{0 \leq j < k} |z_j| \quad \text{with some } r < 1,$$

so that

$$\sum_0^\infty |r_n| \leq C|Au_0^h - A_\lambda u_0^h| \leq 2C|Au_0|.$$

Multiplying (29) by $h(u_n^h - u_{\lambda,n}^h)$ and summing yields

$$(30) \quad \begin{aligned} & \sum_{n=0}^N \left(u_n^h - u_{\lambda,n}^h, \sum_{j=0}^n \gamma_{n-j}(u_j^h - u_{\lambda,j}^h) \right) \\ &= -h \sum_{n=0}^N (u_n^h - u_{\lambda,n}^h, Au_n^h - A_\lambda u_{\lambda,n}^h) + h \sum_{n=0}^N (u_n^h - u_{\lambda,n}^h, r_n). \end{aligned}$$

It is shown in the proof of Theorem 3.1 in [12], that the left-hand side of (30) can be bounded from below by $\delta|u_N^h - u_{\lambda,N}^h|^2$, with $\delta > 0$ depending only on the method (ρ, σ) . We shall consider the right-hand side of (30). Put $v_n \equiv u_0^h$ and use (10). For $n < Th^{-1}$ we then get

$$|u_{\lambda,n}^h - u_0^h| \leq Ch \sum_{j=k}^n |A_\lambda u_0^h| \leq CT|Au_0|,$$

where we also used (i) and (iii). Similarly, we get the uniform boundedness for $\{u_n^h\}$. Hence,

$$(31) \quad h \sum_{n=0}^N (u_n^h - u_{\lambda,n}^h, r_n) \leq CT|Au_0|^2 h, \quad N < Th^{-1}.$$

To bound the first term in the right-hand side of (30) write $J_\lambda x = x - \lambda A_\lambda x$. Since A is monotone, we have $(AJ_\lambda x - Ay, J_\lambda x - y) \geq 0$; and using also $A_\lambda x = AJ_\lambda x$, we get

$$(32) \quad (A_\lambda x - Ay, x - y) \geq \lambda|A_\lambda x|^2 - \lambda(Ay, A_\lambda x) \geq -\frac{\lambda}{2}|Ay|^2.$$

Hence,

$$(33) \quad h \sum_0^N (u_n^h - u_{\lambda,n}^h, Au_n^h - A_\lambda u_{\lambda,n}^h) \geq -\frac{\lambda}{2} h \sum_{n=0}^N |Au_n^h|^2.$$

We show that

$$(34) \quad |Au_n^h| \leq C|Au_0|, \quad n \geq 0, h > 0.$$

Apply (10) with $u_n = u_n^h$ and $v_n = u_{n+1}^h$. Then

$$(35) \quad |u_{n+1}^h - u_n^h| \leq C\{|u_k^h - u_0^h| + h|Au_k^h - Au_0^h|\} \quad \text{for all } n \geq 0.$$

Suppose we normalize: $\alpha_k = 1$ and put $\beta_k = \beta$. Then u_k^h is given by

$$(36) \quad u_k^h + h\beta_k Au_k^h = u_0^h - h(\sigma(1) - \beta_k)Au_0^h \stackrel{\text{def}}{=} v^h.$$

From A -stability (8) it follows that $\beta > 0$. Hence,

$$u_k^h = J_{h\beta} v^h.$$

Thus, using (i)–(iii), and writing $J_\lambda x - x = \lambda A_\lambda x$,

$$\begin{aligned} |u_k^h - u_0^h| &\leq |J_{h\beta} v^h - v^h| + |v^h - u_0^h| \\ &\leq h\beta|A_{h\beta} v^h| + h|\sigma(1) - \beta| |Au_0^h| \\ &\leq h\beta|A_{h\beta} v^h - A_{h\beta} u_0^h| + h\beta|A_{h\beta} u_0^h| + h|\sigma(1) - \beta| |A_{h\beta} u_0^h| \\ &\leq |v^h - u_0^h| + h[2\beta + |\sigma(1)|] |Au_0^h| \\ &\leq |u_0^h - u_0^h| + h|\sigma(1) - \beta| |Au_0^h| + h[2\beta + |\sigma(1)|] |Au_0^h| \\ &\leq h\{1 + 3\beta + 2|\sigma(1)|\} |Au_0^h| = C|Au_0^h|. \end{aligned}$$

From (36) we then also obtain $h\beta_k|Au_k^h| \leq C|Au_0^h|$. Substituting these into (35) gives

$$|u_{n+1}^h - u_n^h| \leq C|Au_0^h|$$

and, hence,

$$|\rho(E)u_n^h| \leq C|Au_0|h \quad \text{for all } n \geq 0.$$

This give us

$$(37) \quad |\sigma(E)Au_n^h| \leq C|Au_0| \quad \text{for all } n \geq 0.$$

But (8) and (9) imply that the roots of $\sigma(\zeta)$ lie strictly inside the unit circle and, therefore, (37) implies (34). Summarizing (30), (31), (33) and (34) we have

$$\begin{aligned} \delta|u_N^h - u_{\lambda, N}^h|^2 &\leq CT|Au_0|^2h + \frac{\lambda}{2}h \sum_0^N |Au_n^h|^2 \\ &\leq CT|Au_0|^2(h + \lambda), \quad N < T/h; \end{aligned}$$

and since $\delta > 0$,

$$(38) \quad |u_n^h - u_{\lambda, n}^h| \leq C|Au_0|\sqrt{[T(h + \lambda)]}, \quad 0 \leq n < T/h; h, \lambda > 0.$$

Combining (25), (26) and (38) we thus have

$$(39) \quad \begin{aligned} |u(nh) - u_n^h| &\leq |u(nh) - u_{\lambda}(nh)| + |u_{\lambda}(nh) - u_{\lambda, n}^h| + |u_{\lambda, n}^h - u_n^h| \\ &\leq C|Au_0|\{\sqrt{(T\lambda)} + h + Th/\lambda + \sqrt{[T(h + \lambda)]}\}, \end{aligned}$$

which then implies (11) and completes the proof. Corollary 1 follows if in (39) we substitute $h^{2/3}$ for $\lambda = \lambda(h)$.

4. Proof of Theorem 3. We shall assume that $u_0 \in D(A)$. The general case follows as in the proof of Theorem 2, since the bounds (19) and (21) are independent of T .

Let $c \in A^{-1}0$ be such that $u(t) \rightarrow c$ (strongly) as $t \rightarrow \infty$. Put $v_n = c$ for all $n \geq 0$ and $u_n = u_{n+N}^h$. Then Theorem 1 gives us

$$(40) \quad |u_n^h - c| \leq C \left\{ \max_{0 \leq j < k} [|u_{N+j}^h - c| + h|Au_{N+j}^h|] \right\} \quad \text{for } n \geq N.$$

Together with (34) this gives

$$(41) \quad |u_n^h - c| \leq C \max_{0 \leq j < k} |u_{N+j}^h - c| + Ch|Au_0| \quad \text{for } n \geq N.$$

Fix $\epsilon > 0$ and take $T < \infty$ so that

$$(42) \quad |u(t) - c| < \epsilon \quad \text{for } t \geq T.$$

Choose $h_0 < 1/k$ such that, by Theorem 2,

$$(43) \quad |u(nh) - u_n^h| < \epsilon \quad \text{for } nh < T + 2 \text{ for } h < h_0.$$

Put $N = [T/h] + 1$. For $mh > T$ we then have

$$\begin{aligned} |u(mh) - u_m^h| &\leq |u(mh) - c| + |c - u_m^h| \\ &< \epsilon + C \max_{0 \leq j < k} |u_{N+j}^h - c| + Ch|Au_0|. \end{aligned}$$

But using (42) and (43), we get

$$|u_{N+j}^h - c| \leq |u_{N+j}^h - u((N+j)h)| + |u((N+j)h) - c| < 2\epsilon;$$

and hence,

$$(44) \quad |u(mh) - u_m^h| < (2C + 1)\epsilon + Ch|Au_0|, \quad mh > T;$$

and the conclusion follows.

5. Proof of Theorem 4. For the trapezoidal rule we do not have a bound of the form (10); but we shall overcome this difficulty by considering a closely related method, the implicit midpoint rule

$$(45) \quad h^{-1}(\hat{u}_{n+1}^h - \hat{u}_n^h) + A \left(\frac{\hat{u}_{n+1}^h + \hat{u}_n^h}{2} \right) = 0 \quad \text{for } n \geq 0.$$

Existence and uniqueness of a solution $\{\hat{u}_n^h\}$ is obtained as follows: Write (45) in the form

$$(46) \quad v_n + \frac{h}{2}Av_n = \hat{u}_n^h,$$

with \hat{u}_n^h given and $v_n (= \frac{1}{2}(\hat{u}_{n+1}^h + \hat{u}_n^h))$ unknown. But then $v_n = J_{h/2}\hat{u}_n^h \in D(A)$ and the process may be continued. Further,

$$v_{n+1} - v_n = \frac{\hat{u}_{n+2}^h - \hat{u}_n^h}{2} = -\frac{h}{2}(Av_{n+1} + Av_n)$$

so that $\{v_n\}$ satisfies (15) with the initial value $v_0 = J_{h/2}\hat{u}_0^h \in D(A)$. On the other hand, (15) has a unique solution $\{u_n^h\} \subset D(A)$ for all $u_0^h \in D(A)$ since

$$u_{n+1}^h = J_{h/2} \left(u_n^h + \frac{h}{2}Au_n^h \right) \in D(A) \quad \text{for all } n \geq 0.$$

Hence we have reduced our problem to showing that, with $\hat{u}_0^h = u_0$,

$$\sup_{0 \leq n < T/h} |u(nh) - \hat{u}_n^h| \rightarrow 0 \quad \text{as } h \downarrow 0;$$

and that if (13) holds, then

$$\sup_{n \geq 0} |u(nh) - \hat{u}_n^h| \rightarrow 0 \quad \text{as } h \downarrow 0.$$

Convergence for

$$v_n = (\hat{u}_{n+1}^h + \hat{u}_n^h)/2$$

will follow since $u(t)$ is uniformly continuous on compact sets $[0, T]$ and, assuming (13), on the half line $t \geq 0$.

For notational simplicity, consider the solutions $\{x_n\}$, $\{y_n\}$ of the equations

$$(47) \quad h^{-1}(x_{n+1} - x_n) + A\left(\frac{x_{n+1} + x_n}{2}\right) = 0,$$

$$(48) \quad h^{-1}(y_{n+1} - y_n) + A\left(\frac{y_{n+1} + y_n}{2}\right) = 0.$$

Forming the difference of (47) and (48), multiplying it by $\frac{1}{2}(x_{n+1} - y_{n+1} - x_n + y_n)$, and using the monotonicity of A yields

$$(49) \quad |x_{n+1} - y_{n+1}| \leq |x_n - y_n|, \quad n \geq 0.$$

Applying (49) with $y_n = x_{n+1}$ gives

$$(50) \quad \left| A\left(\frac{x_{n+1} + x_n}{2}\right) \right| \leq h^{-1}|x_1 - x_0| = |A_{h/2}x_0| \leq |Ax_0|.$$

Denote by $x_{\lambda,n}$ the solution of (47) with A replaced by A_λ and $x_{\lambda,0} = x_0$. Multiplying

$$x_{n+1} - x_{\lambda,n+1} - x_n + x_{\lambda,n} + h \left\{ A\left(\frac{x_{n+1} + x_n}{2}\right) - A_\lambda\left(\frac{x_{\lambda,n+1} + x_{\lambda,n}}{2}\right) \right\} = 0$$

by $\frac{1}{2}\{x_{n+1} - x_{\lambda,n+1} + x_n - x_{\lambda,n}\}$, we get, using (32), and (50),

$$|x_n - x_{\lambda,n}| \leq |Ax_0|\sqrt{(\lambda nh)} \quad \text{for } n \geq 0,$$

that is,

$$(51) \quad |\hat{u}_n^h - \hat{u}_{\lambda,n}^h| \leq |Au_0|\sqrt{(\lambda nh)} \quad \text{for } n \geq 0; \lambda, h > 0.$$

Since the implicit midpoint rule is a convergent method (for Lipschitz-continuous problems on compact intervals $[0, T]$) the proof of the reduced problem can now be completed as in the earlier chapters, using (51), and (49) or

$$(52) \quad |\hat{u}_{n+1}^h - \hat{v}_{n+1}^h| \leq |\hat{u}_n^h - \hat{v}_n^h|$$

in place of (38) and (10).

Remark. The implicit midpoint rule is the one-leg method associated with the trapezoidal rule, [6]. A generalization of Theorem 4 to all G -stable [6] methods is straightforward but since the trapezoidal rule is the only practically important method which does not satisfy (9) we do not want to introduce the machinery needed.

However, if the conjecture that A -stability implies G -stability (see [7]) is correct, then some parts of the proof of Theorem 2 could be simplified. The analogue of Theorem 1 for G -stable one-leg methods has been given in [11], with a more direct proof than in [10].

6. Proof of Theorem 5. Put $a = -1/r$, where r is given by (16). Assume that $(I + ahA_\lambda)^{-1}$ is Lipschitz-continuous with constant $L = L(\lambda, h) < \infty$ and that $A_\lambda(I + ahA_\lambda)^{-1} - \alpha I$ is maximally monotone with $\alpha = \alpha(\lambda, h) > -\infty$. Then we obtain, using line (3.23) in [12], that for $C, b > 0$ depending only on (ρ, σ) ,

$$(53) \quad |u_{\lambda,n}^h - u_\lambda(nh)| \leq CL(\lambda, h)(1 - |\alpha(\lambda, h)|hb)^{-(1+T)/2h} \cdot \left\{ h \sum_{n=k}^{T/h} |\tau_{\lambda,n}^h| + \max_{0 \leq j < k} [|u_0 - u_\lambda(jh)| + h|A_\lambda u_0 - A_\lambda u_\lambda(jh)|] \right\}.$$

From (23) we obtain $|\tau_{\lambda,n}^h| \leq C(h/\lambda)|Au_0|$ and, therefore,

$$h \sum_{n=k}^{T/h} |\tau_{\lambda,n}^h| \leq CT|Au_0|(h/\lambda).$$

Since $|A_\lambda u_\lambda(t)|$ is nonincreasing in t , we also have

$$\begin{aligned} \max_{0 \leq j < k} [|u_0 - u_\lambda(jh)| + h|A_\lambda u_0 - A_\lambda u_\lambda(jh)|] \\ \leq (k - 1)h|A_\lambda u_0| + h|A_\lambda u_0| + h|A_\lambda u_0| \leq (k + 1)h|Au_0|. \end{aligned}$$

Thus, (53) is of the form

$$(54) \quad |u_\lambda(nh) - u_{\lambda,n}^h| \leq CL(\lambda, h) \{1 - |\alpha(\lambda, h)|bh\}^{-(1+T)/2h} |Au_0| \left\{ h + \frac{hT}{\lambda} \right\}.$$

Since A_λ is Lipschitz-continuous with constant $1/\lambda$, we have $L(\lambda, h) \leq (1 - |a|h/\lambda)^{-1} \rightarrow 1$ if $h/\lambda \rightarrow 0$. To estimate $\alpha(\lambda, h)$ from below, put $x = (I + ahA_\lambda)^{-1}u$ and $y = (I + ahA_\lambda)^{-1}v$. Then

$$\begin{aligned} (A_\lambda(I + ahA_\lambda)^{-1}u - A_\lambda(I + ahA_\lambda)^{-1}v, u - v) \\ = (A_\lambda x - A_\lambda y, x - y) + ah(A_\lambda x - A_\lambda y, A_\lambda x - A_\lambda y) \\ \geq ah|A_\lambda x - A_\lambda y|^2 \geq (ah/\lambda^2)|x - y|^2 \geq ah(L(\lambda, h)/\lambda)^2|u - v|^2, \end{aligned}$$

so that

$$|\alpha(\lambda, h)| \leq |a|h \left(\frac{L(\lambda, h)}{\lambda} \right)^2 \lesssim |a|/c$$

if $\lambda \downarrow 0$ with $\lambda^2 \geq ch$; and we conclude that $\{1 - |\alpha(\lambda, h)|bh\}^{-(1/2h)(1+T)}$ remains bounded for λ small, $\lambda < \lambda_0$ say.

Summarizing (54) and (26), we have for $nh < T$,

$$\begin{aligned} |u(nh) - u_{\lambda,n}^h| &\leq |u(nh) - u_\lambda(nh)| + |u_\lambda(nh) - u_{\lambda,n}^h| \\ &\leq C(\lambda_0, T) |Au_0| \{\lambda^{1/2} + h + h/\lambda\}, \end{aligned}$$

which completes the proof.

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