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Some Primes with Interesting Digit Patterns

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Abstract. Several tables of prime numbers whose forms are generalizations of the form $(10^n - 1)/9$ of the repunit numbers are presented. The repunit number $(10^{317} - 1)/9$ is shown to be a prime.

1. Introduction. Considerable interest has been expressed over a long period of time in the repunit numbers. These are numbers which in decimal notation are made up only of the unit digit, i.e. numbers of the form $(10^n - 1)/9$. Surveys of the literature are given in Yates [8], [9]. In this paper we consider some integers whose forms are generalizations of that of a repunit number and we tabulate some primes of these various forms. We also present a new repunit prime.

2. The Extended Forms. The numbers which we discuss in this paper have the following forms:

$$N_1(n, r) = (10^{n+1} + 9r - 10)/9,$$

$$N_2(n, r) = ((9r + 1)10^n - 1)/9,$$

$$N_3(n, r) = (10^{2n+1} + 9(r - 1)10^n - 1)/9,$$

$$N_4(n, k, r) = 10^r B_k(10^{nk} - 1)/(10^k - 1) + B_r \qquad (1 \leqslant r \leqslant k),$$

where $n \geqslant 1$, $1 \leqslant r \leqslant 9$, and

$$B_k = (10^{k+1} - 9k - 10)/81 \qquad (1 \leqslant k \leqslant 9).$$

Note that each of these is a generalization of the repunit numbers ($r = 1$ and $k = 1$), although we do not consider these numbers, themselves, here. We illustrate the digital pattern for N_1, N_2, N_3, N_4 below:

$$N_1(n, r) = \underbrace{111 \cdots 1}_{n \text{ ones}}r,$$

$$N_2(n, r) = r\underbrace{111 \cdots 1}_{n \text{ ones}},$$

$$N_3(n, r) = \underbrace{111 \cdots 1}_{n \text{ ones}}r\underbrace{111 \cdots 1}_{n \text{ ones}},$$

$$N_4(n, r, k) = \underbrace{B_kB_kB_k \cdots B_k}_{n\ B_k\text{'s}}B_r,$$

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where B_k $(1 \leqslant k \leqslant 9)$ is a block of digits with the form

$$123 \cdots k.$$

For example,

$$N_1(2, 3) = 113, \quad N_2(2, 3) = 311, \quad N_3(2, 3) = 11311,$$
$$N_4(2, 5, 3) = 1234512345123.$$

Note that $N_1(n, r)$ cannot be a prime for $r = 2, 4, 5, 6, 8$. Also, $N_3(n, 2)$ cannot be a prime, as

$$N_3(n, 2) = (10^n + 1)(10^{n+1} - 1)/9.$$

3. The Tables. Primes of the form N_1, N_2, N_3, N_4, which have no more than 100 digits were found. This was done on an IBM system 370 model 168 computer, using the prime testing routines described in Brillhart, Lehmer, and Selfridge [2], Williams and Judd [5], [6].

Naturally these routines were used only on those numbers N_1, N_2, N_3, N_4, which have no small prime (< 1000) factors. The results of these computations are presented in Tables 1, 2, 3 and 4.

Primes of the form $N_1(n, r)$ $(n \leqslant 99)$ are given in Table 1.

r	$n \leqslant 99$
3	1, 2, 4, 8, 10, 23
7	1, 3, 4, 7, 22, 28, 39
9	1, 4, 5, 7, 16, 49

TABLE 1

In Table 2 we give all those primes of the form $N_2(n, r)$ such that $n \leqslant 99$.

r	$n \leqslant 99$
2	2, 3, 12, 18, 23, 57
3	1, 2, 5, 10, 11, 13, 34, 47, 52, 77, 88
4	1, 3, 13, 25, 72
5	5, 12, 15, 84
6	1, 5, 7, 25, 31
7	1, 7, 55
8	2, 3, 26
9	2, 5, 20, 41, 47, 92

TABLE 2

In Table 3 we give all those primes of the form $N_3(n, r)$ such that $n \leqslant 50$.

r	$n \leqslant 50$	r	$n \leqslant 50$
3	1, 2, 19	7	3, 33
4	2, 3, 32, 45	8	1, 4, 6, 7
5	1, 7, 45	9	1, 4, 26
6	10, 14, 40		

TABLE 3

Finally, in Table 4, we give all the primes of the form $N_4(n, k, r)$ for $m = nk + r \leqslant 100$.

k	$m = nk + r \leqslant 100$	k	$m \leqslant 100$
2	7, 11, 43	6	Nil
3	4, 7, 52	7	Nil
4	55, 71	8	95
5	21	9	10, 28, 70

TABLE 4

Remarks. (1) The number $N = N_1(83, 3)$ is a base 13 pseudoprime but has not yet been shown a prime. Using the notation of [6] and a factor bound $B = 5988337680$, D. H. Lehmer found that $F_1 = 2^3 \cdot 1531$, $F_2 = 2 \cdot 3$, $F_4 = 2 \cdot 5 \cdot 2069 \cdot 215789$, $F_3 = 7 \cdot 14869$, $F_6 = 3 \cdot 271$. All the cofactors are composite. Even using the extended methods of Williams and Holte [7], this is still not enough information to demonstrate the primality of N.

(2) The prime $N_4(1, 9, 1)$ had been previously discovered by Madachy [4] and the prime $N_4(3, 9, 1)$ by Finkelstein and Leybourne [3].

4. A New Repunit Prime. Because of the interest in repunit primes, all integers of the form $R_n = (10^n - 1)/9$ ($n \leqslant 1000$, n prime) not divisible by a small prime were tested for pseudoprimality. It was discovered that R_n is a base 13 pseudoprime only for $n = 2$, 19, 23 and 317. It is well known that R_2, R_{19} and R_{23} are primes but the result for $n = 317$ was surprising, especially in view of the work of Brillhart and Selfridge [1]. In [1] it is stated that R_n is not a base 3 pseudoprime for any prime n such that $29 \leqslant n \leqslant 359$. R_{317} was tested again for pseudoprimality with 3 different programs and 50 different prime bases. In each case R_{317} turned out to be a pseudoprime.

The number R_{317} is now known to be a prime. We give below the method which was used to demonstrate this. The algorithm used is essentially that of [2].

We first note that

$$R_{317} - 1 = (10^{317} - 10)/9$$
$$= 10 \cdot (10^{79} - 1)/9 \cdot (10^{79} + 1) \cdot (10^{158} + 1).$$

All the prime factors of $(10^{79} - 1)/9$ which are less than 10^8 are 317, 6163, 10271, 307627 and the cofactor is composite. Also, all the prime factors of $10^{158} + 1$ which are less than 6000 are 101 and 5689. Finally, all the prime factors of $10^{79} + 1$ which are less than 10^8 are 11, 1423 and the cofactor is composite. Fortunately, John Brillhart was able to supply the additional factor 9615060929* of the cofactor; and the resulting cofactor after division by this prime is the 65 digit number

$$M = 66443174541490579097997510158021076958392938976011506949065646573.$$

This number was found to be a base 13 pseudoprime and was, subsequently, proved prime by using the methods of [5] with a factor bound of 2×10^6 and the data

$$F_1 = 2^2 \cdot 79, \quad F_2 = 2 \cdot 3 \cdot 61 \cdot 157 \cdot 199^2,$$
$$F_4 = 2 \cdot 5 \cdot 29 \cdot 149 \cdot 421 \cdot 541 \cdot 2137,$$
$$F_3 = 19 \cdot 2671 \cdot 2719, \quad F_6 = 2 \cdot 3 \cdot 13 \cdot 1411783.^{**}$$

Using all the prime factors of $R_{317} - 1$ found above, we have a completely factored part F of $R_{317} - 1$ which exceeds 3.54×10^{101}. The tests of [2] were used with a factor bound of 10^8 on $10^{79} - 1$ and 6000 on $10^{158} + 1$. These tests were satisfied and it follows that if p is any prime divisor of R_{317}, then

$$p \equiv 1 \pmod{q_1 q_2 F},$$

where q_1, q_2 are prime divisors of the cofactors of $10^{79} - 1$ and $10^{158} + 1$, respectively. Hence, $p > 2.12 \times 10^{113}$ and, if R_{317} is composite, it must be the product of only two primes p_1, p_2 with

$$p_1 = m_1 F + 1, \quad p_2 = m_2 F + 1.$$

If this is the case, we have

$$R_{317} - 1 = p_1 p_2 - 1 = m_1 m_2 F^2 + (m_1 + m_2)F;$$

thus,

$$m_1 + m_2 \equiv (R_{317} - 1)/F \pmod{F}.$$

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*This factor was found by S. Wagstaff.

**D. H. Lehmer has found, using the ILLIAC IV, that with the factor bound increased to 4.2×10^9 only F_2 increases. It becomes

$$2 \cdot 3 \cdot 61 \cdot 157 \cdot 199^2 \cdot 20373173 \cdot 2220165587 \cdot 2746999987.$$

This by itself is sufficient, using only the methods of [2], to prove M a prime.

He also found, independently, Wagstaff's factor of $10^{79} + 1$ by searching for prime factors of $10^{79} - 1$, $10^{79} + 1$ and $10^{158} + 1$ to a factor bound of $3 \circ 10^{11}$. In spite of the fact that the cofactors of $10^{79} - 1$ and $10^{158} + 1$ are both composite, no factors were found other than those given above.

Now $(R_{317} - 1)/F \equiv X \pmod{F}$, where $X > 3.53 \times 10^{101}$ and $X < F$ and one of m_1 or m_2 must exceed $\frac{1}{2}X$. Further

$$R_{317} > m_1 m_2 F^2 > \tfrac{1}{2} q_1 q_2 X F^2 > 1.32 \times 10^{316} > R_{317}.$$

As this is impossible, R_{317} can only be a prime.

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