MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 144
OCTOBER 1978, PAGES 1241–1243

Finite Differences of the Partition Function

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Abstract. From the Hardy-Ramanujan-Rademacher formula for $p(n)$—the number of unrestricted partitions of n, it is not difficult to deduce that there exists a least positive integer $n_0(r)$ such that $V^r p(n) \geqslant 0$ for each $n \geqslant n_0(r)$, where $Vp(n) = p(n) - p(n-1)$ and $V^r p(n) = V\{V^{r-1}p(n)\}$. In this note, we give values of $n_0(r)$ for each $r \leqslant 10$ and conjecture that $n_0(r)/r^3 \sim 1$.

1. Notation. In the following,

small letters denote positive integers unless stated otherwise;

$p(n)$ denotes the number of unrestricted partitions of n;

$p(n, m)$ is the number of partitions of n into exactly m summands, when $m \leqslant n$; and we take as usual

$$p(0) = 1, \qquad p(-n) = 0;$$

$$p(n, m) = 0 \quad \text{for } n < m; \qquad p(0, m) = 0 = p(-n, m).$$

For any arithmetic function $f(n)$, the operator V is defined by

$$Vf(n) = f(n) - f(n-1) \quad \text{and} \quad V^r f(n) = V\{V^{r-1}f(n)\}.$$

2. Differences of $p(n)$. We have [1]

$$p(n) - p(n-1) = \sum_{m \geqslant 1} p(n-m, m) \quad \text{for each } n \geqslant 1; \tag{1}$$

so that $Vp(n) \geqslant 0$ for $n \geqslant 1$. For $n = 0$, $Vp(n) = 1$. Again,

$$\begin{aligned} V^2 p(n) &= p(n) - 2p(n-1) + p(n-2) \\ &= \sum_{m \geqslant 1} \{p(n-m, m) - p(n-1-m, m)\}, \qquad n \geqslant 2. \end{aligned} \tag{2}$$

Hence, we have the known result

$$V^2 p(n) \geqslant 0 \quad \text{for } n \geqslant 2.$$

For $n = 1$, however, $V^2 p(n) = -1$. For $n = 0$, $V^2 p(n) = 1$.

Using the well-known Hardy-Ramanujan-Rademacher series for $p(n)$, it is not difficult to show that

$$V^r p(n) = C_r p(n)(1 + O(n^{-1/2})), \tag{3}$$

Received November 28, 1977.
AMS (MOS) subject classifications (1970). Primary 10A45, 12H10.

where $C_r = (\pi/\sqrt{6})^r/4\sqrt{3}$. Hence, there exists a least positive integer $n_0(r)$ such that

$$V^r p(n) \geqslant 0 \quad \text{for each } n \geqslant n_0(r).$$

More explicitly, on the basis of our calculations, we can say that

for each odd $n < n_0(r)$, $V^r p(n)$ is negative;

for each odd $n \geqslant n_0(r)$, $V^r p(n)$ is $\geqslant 0$; while

for each even $n \geqslant 0$, $V^r p(n) \geqslant 0$.

r	n_0	n	$V^r p(n)$	r	n_0	n	$V^r p(n)$
3	26	21	−4	7	352	349	−780 36820
		23	−2			351	−424 37469
		25	−4			352	17748 68363
		26	32			353	8 65716
		27	1			355	522 81173
		28	38			359	1841 78679
		29	5				
4	68	65	−87	8	510	509	−57339 70174
		67	−64			510	33 48946 29181
		68	1497			511	12505 02420
		69	17			513	93196 02052
		71	152				
5	134	129	−8840	9	704	703	−45 72279 29371
		133	−3143			704	7839 27672 53289
		134	1 12115			705	99 97628 46394
		135	951			709	451 37612 23991
6	228	223	−7 89593	10	934	933	−14518 50404 20380
		225	−5 59660			934	20 81467 28166 39740
		227	−2 47781			935	19110 28378 57344
		228	123 79258			937	56641 87086 56258
		229	1 25723				

3. The Table. In the table above, we give the values of $n_0(r)$ for $3 \leqslant r \leqslant 10$. We give also values of $V^r p(n)$ for some values of n in the neighborhood of $n_0(r)$ to bring out clearly how the change takes place. The Royal Society *Tables of Partitions* [2] were freely used in preparing this table.

It is noteworthy that within the limits of our table

$$n_0(r)/r^3 \text{ is about } 1.$$

We conjecture that

$$n_0(r)/r^3 \sim 1.$$

We might here mention that the problem discussed in this note was raised by George E. Andrews.

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2. H. GUPTA, C. E. GWYTHER & J. C. P. MILLER, *Tables of Partitions*, University Press, Cambridge, 1958.