Finite Differences of the Partition Function

By Hansraj Gupta

Abstract. From the Hardy-Ramanujan-Rademacher formula for \( p(n) \) — the number of unrestricted partitions of \( n \), it is not difficult to deduce that there exists a least positive integer \( n_0(r) \) such that \( V^r p(n) > 0 \) for each \( n \geq n_0(r) \), where \( Vp(n) = p(n) - p(n - 1) \) and \( V^r p(n) = V\{ V^{r-1} p(n) \} \). In this note, we give values of \( n_0(r) \) for each \( r < 10 \) and conjecture that \( n_0(r)/r^3 \sim 1 \).

1. Notation. In the following, small letters denote positive integers unless stated otherwise; \( p(n) \) denotes the number of unrestricted partitions of \( n \); \( p(n, m) \) is the number of partitions of \( n \) into exactly \( m \) summands, when \( m \leq n \); and we take as usual

\[
p(0) = 1, \quad p(-n) = 0;
\]

\[
p(n, m) = 0 \quad \text{for } n < m; \quad p(0, m) = 0 = p(-n, m).
\]

For any arithmetic function \( f(n) \), the operator \( V \) is defined by

\[
Vf(n) = f(n) - f(n - 1) \quad \text{and} \quad V^r f(n) = V\{ V^{r-1} f(n) \}.
\]

2. Differences of \( p(n) \). We have [1]

\[
p(n) - p(n - 1) = \sum_{m \geq 1} p(n - m, m) \quad \text{for each } n \geq 1;
\]

so that \( Vp(n) > 0 \) for \( n \geq 1 \). For \( n = 0 \), \( Vp(n) = 1 \). Again,

\[
V^2 p(n) = p(n) - 2p(n - 1) + p(n - 2)
\]

\[
= \sum_{m \geq 1} \{ p(n - m, m) - p(n - 1 - m, m) \}, \quad n \geq 2.
\]

Hence, we have the known result

\[ V^2 p(n) > 0 \quad \text{for } n \geq 2. \]

For \( n = 1 \), however, \( V^2 p(n) = -1 \). For \( n = 0 \), \( V^2 p(n) = 1 \).

Using the well-known Hardy-Ramanujan-Rademacher series for \( p(n) \), it is not difficult to show that

\[
V' p(n) = C_r p(n)(1 + O(n^{-1/2}))
\]
where \( C_r = (\pi/\sqrt{6})^{r}/4\sqrt{3} \). Hence, there exists a least positive integer \( n_0(r) \) such that

\[ V^r p(n) \geq 0 \quad \text{for each} \quad n \geq n_0(r). \]

More explicitly, on the basis of our calculations, we can say that

- for each odd \( n < n_0(r) \), \( V^r p(n) \) is negative;
- for each odd \( n \geq n_0(r) \), \( V^r p(n) \) is \( \geq 0 \); while
- for each even \( n \geq 0 \), \( V^r p(n) \) \( \geq 0 \).

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3. The Table. In the table above, we give the values of \( n_0(r) \) for \( 3 \leq r \leq 10 \). We give also values of \( V'(n) \) for some values of \( n \) in the neighborhood of \( n_0(r) \) to bring out clearly how the change takes place. The Royal Society Tables of Partitions [2] were freely used in preparing this table.

It is noteworthy that within the limits of our table

\[
n_0(r)/r^3 \text{ is about } 1.
\]

We conjecture that

\[
n_0(r)/r^3 \sim 1.
\]

We might here mention that the problem discussed in this note was raised by George E. Andrews.

Panjab University
Chandigarh 160014, India

402 Mumfordganj
Allahabad 211002, India
