

## Interpolation by Convex Quadratic Splines\*

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**Abstract.** Algorithms are presented for computing a quadratic spline interpolant with variable knots which preserves the monotonicity and convexity of the data. It is also shown that such a spline may not exist for fixed knots.

**1. Introduction.** Interpolation to convex data by a convex polynomial spline has been investigated by Passow and Roulier in [6]. McAllister, Passow, and Roulier [5] present an efficient computational algorithm for such interpolation. In both of these papers, the knots of the spline are the interpolation points. It is shown in [6] that the degree of the various piecewise polynomials may be forced to be arbitrarily high by a suitable choice of data points. The purpose of this paper is to show that in convex interpolation with such splines, this undesirable property can occur for any choice of fixed knots. On the other hand, we will present an algorithm for interpolation by convex quadratic splines with knots at the interpolation points and at most one additional properly chosen knot between each interpolation point.

We note that elimination of undesirable oscillations (without increasing the degree of the polynomial interpolant), by the addition of knots, is studied in de Boor [1, Chapter 16], and in Dube ([2] and [3]). Their methods are essentially different from those presented here.

We also note that our algorithm will only add knots when required, that is, when the data bends very sharply. In this case the resulting curve though differentiable will look very much like a curve with corners. This, of course, is due to the fact that any differentiable shape preserving function interpolating this data would behave in this way.

**2. Notation and Preliminary Concepts.** Let  $\Delta = \{x_0, x_1, \dots, x_N\}$  be a fixed set of real numbers with  $x_i < x_{i+1}$ ,  $i = 0, \dots, N-1$ . For  $j \leq n$  the set of *splines of degree  $n$  and deficiency  $n - j$  on  $\Delta$*  is denoted by  $S_n^j(\Delta)$ . Thus,  $f \in S_n^j(\Delta)$  if and only if  $f \in C^j[x_0, x_N]$  and  $f$  is an algebraic polynomial of degree  $n$  or less on  $[x_{i-1}, x_i]$  for  $i = 1, 2, \dots, N$ .

Let  $y_0, y_1, \dots, y_N$  be real numbers and define the slopes  $S_i$  by

$$(2.1) \quad S_i = (y_i - y_{i-1}) / (x_i - x_{i-1}) \quad \text{for } i = 1, 2, \dots, N.$$

Then, the data  $\{(x_i, y_i)\}_{i=0}^N$  are said to be *nondecreasing* if  $y_0 \leq y_1 \leq \dots \leq y_N$  and *nonconcave* if  $S_1 \leq S_2 \leq \dots \leq S_N$ . If no equality exists in these two cases, we say that the data are *increasing* or *convex*, respectively.

The following definition is a special case of a definition given in [5] and [6].

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DEFINITION 2.1. Suppose the data  $\{(x_i, y_i)\}_{i=0}^N$  are nondecreasing and nonconcave. Let  $\bar{x}_i = x_{i-1} + \Delta x_i/2$  where  $\Delta x_i = x_i - x_{i-1}$  for  $i = 1, 2, \dots, N$ . A set of numbers  $\{t_i\}_{i=1}^N$  is said to be  $\frac{1}{2}$ -admissible if the piecewise linear function  $L(x)$  generated by the points  $(x_0, y_0), (\bar{x}_1, t_1), (\bar{x}_2, t_2), \dots, (\bar{x}_N, t_N), (x_N, y_N)$  passes through the points  $(x_i, y_i), i = 1, \dots, N - 1$ , and is nondecreasing and nonconcave.

The following theorem is a special case of the theorem of Passow and Roulier [6].

THEOREM 2.1. *Given nondecreasing, nonconcave data  $\{(x_i, y_i)\}_{i=0}^N$  there exists a quadratic spline  $f$  (i.e.,  $f \in S_2^1(\Delta)$ ) satisfying*

$$(2.2) \quad f(x_i) = y_i \quad \text{for } i = 0, 1, \dots, N,$$

$$(2.3) \quad f'(x) \geq 0 \quad \text{on } [x_0, x_N],$$

and

$$(2.4) \quad f'(x) \text{ is nondecreasing on } [x_0, x_N]$$

if, and only if, there exists a set of  $\frac{1}{2}$ -admissible points  $\{t_i\}_{i=1}^N$  for these data.

(It is actually shown in [6] that if such a set of  $\frac{1}{2}$ -admissible points exists then for any positive integer  $m$  there is  $f \in S_{2m}^m(\Delta)$  satisfying (2.2), (2.3) and (2.4). We will restrict ourselves to  $S_2^1(\Delta)$ .) Furthermore, if such  $\frac{1}{2}$ -admissible points exist, then such an  $f$  can be constructed as follows:

Let  $L$  be the piecewise linear function described in definition (2.1). For  $i = 1, 2, \dots, N$  define

$$(2.5) \quad q_i(x) = \frac{1}{(\Delta x_i)^2} [L(x_{i-1})(x_i - x)^2 + 2L(\bar{x}_i)(x - x_{i-1})(x_i - x) + L(x_i)(x - x_{i-1})^2].$$

We note that  $q_i$  is the 2nd degree Bernstein polynomial of  $L$  on  $[x_{i-1}, x_i]$ . We define  $f \in S_2^1(\Delta)$  satisfying (2.2), (2.3), and (2.4) by

$$f(x) = q_i(x) \quad \text{on } [x_{i-1}, x_i], \quad i = 1, 2, \dots, N.$$

We now describe a special case of an algorithm developed by McAllister, Passow, and Roulier [5] which gives necessary and sufficient conditions for the existence of such  $\frac{1}{2}$ -admissible points.

*The  $\frac{1}{2}$ -algorithm.* Define  $m_0 = 0$  and  $M_0 = S_1$ . Now for  $i = 1, 2, \dots, N - 1$  define

$$m_i = 2S_i - M_{i-1} \quad \text{and} \quad M_i = \min(S_{i+1}, 2S_i - m_{i-1}).$$

The following then is a special case of a theorem in [5].

THEOREM 2.2. *Given nondecreasing, nonconcave data  $(x_i, y_i), i = 0, 1, \dots, N$ ,*

(2.6) *there exist  $\frac{1}{2}$ -admissible points for these data*

if and only if

(2.7) *the  $\frac{1}{2}$ -algorithm can be completed with  $m_i \leq S_{i+1}$  for  $i = 1, 2, \dots, N - 1$ .*

If (2.7) holds, then the construction of a piecewise linear function  $L$  as described above can be accomplished using  $m_{N-1}$  and  $M_{N-1}$ . We simply take  $t_N$  to be any point which lies between

$$a_N = y_{N-1} + m_{N-1}(\bar{x}_N - x_{N-1}) = y_{N-1} + m_{N-1} \left( \frac{x_N - x_{N-1}}{2} \right)$$

and

$$b_N = y_{N-1} + M_{N-1}(\bar{x}_N - x_{N-1}) = y_{N-1} + M_{N-1} \left( \frac{x_N - x_{N-1}}{2} \right).$$

The numbers  $t_{N-1}, t_{N-2}, \dots, t_1$  are then uniquely determined by the choice of  $t_N$ . We note furthermore that if  $m_{N-1} = M_{N-1} \leq S_N$ , then the interpolatory quadratic spline satisfying (2.2), (2.3), and (2.4) is unique.

**3. Fixed Knots.** In the previous section, the numbers  $x_i, i = 0, 1, \dots, N$ , were the abscissas of data points, and the mesh  $\Delta$  was the collection of all such abscissas. In this section  $x_i, i = 0, 1, \dots, N$ , will stand for the abscissas of data points, but  $\Delta$  will now be an arbitrary finite collection of real numbers which will be the knots of the piecewise polynomials. We will show that, given any mesh  $\Delta$ , any four such  $x_i, i = 0, 1, 2, 3$ , and any  $n$ , there exists a set of four increasing convex data points with these  $x_i$  as abscissas such that no  $f \in S_n^1(\Delta)$  can interpolate the data and be convex and increasing.

**THEOREM 3.1.** *Let the integer  $n \geq 1$ , real numbers  $x_0 < x_1 < x_2 < x_3$  and mesh  $\Delta$  be given. There exist numbers  $y_0, y_1, y_2, y_3$  such that the data  $(x_i, y_i), i = 0, 1, 2, 3$ , are increasing and convex and such that no  $f \in S_n^1(\Delta)$  satisfies*

$$(3.1) \quad f(x_i) = y_i \quad \text{for } i = 0, 1, 2, 3,$$

and

$$(3.2) \quad f \text{ is convex and increasing on } [x_0, x_3].$$

The following lemma will be helpful in the proof of Theorem 3.1: Let  $\Delta$  be the mesh described above. We may assume without loss of generality that  $x_i \in \Delta, i = 0, 1, 2, 3$ . Let

$$l = \max\{x | x \in \Delta \text{ and } x < x_1\} \quad \text{and} \quad u = \min\{x | x \in \Delta \text{ and } x > x_1\}.$$

By  $\|\cdot\|_{a,b}$  we mean the supremum norm on the interval  $[a, b]$ . That is, if  $f \in C[a, b]$ , then

$$\|f\|_{a,b} = \sup\{|f(x)|; x \in [a, b]\}.$$

**LEMMA 3.1.** *Let the mesh  $\Delta$  be given and assume that  $(x_i, y_i), i = 0, 1, 2, 3$ , are convex and increasing. If  $f \in S_n^1(\Delta)$  satisfies (3.1) and (3.2), then*

$$(3.3) \quad n^2 \geq \frac{S_2}{(x_2 - x_1)(S_3 - S_2)/(u - x_1) + (x_1 - x_0)S_1/(x_1 - l)},$$

where the numbers  $S_i$  are defined in (2.1), that is,

$$S_i = (y_i - y_{i-1})/(x_i - x_{i-1}) \quad \text{for } i = 1, 2, 3.$$

*Proof.* Observe that on  $[l, x_1]$  we have

$$0 \leq f(x) - y_0 \leq y_1 - y_0 \quad \text{and} \quad \left| f(x) - \frac{y_1 + y_0}{2} \right| \leq \frac{y_1 - y_0}{2}.$$

Thus, by Markov's inequality (see [4, p. 40]) applied to  $f(x) - (y_1 + y_0)/2$  we have, for  $\epsilon = x_1 - l$ ,

$$\|f'\|_{l, x_1} \leq \frac{y_1 - y_0}{\epsilon} n^2 = \frac{S_1(x_1 - x_0)}{\epsilon} n^2.$$

Since  $f$  is convex and increasing, we have

$$(3.4) \quad S_1 \leq f'(x_1) \leq S_1 \frac{(x_1 - x_0)}{\epsilon} n^2.$$

Let  $h$  be the straight line passing through  $(x_1, y_1)$  and  $(x_2, y_2)$ , and let  $k$  be the straight line passing through  $(x_2, y_2)$  and  $(x_3, y_3)$ . It is easy to see that

$$0 \leq h(x) - f(x) \leq h(x) - k(x) \leq (x_2 - x_1) (S_3 - S_2)$$

and

$$\left| h(x) - f(x) - \frac{(x_2 - x_1) (S_3 - S_2)}{2} \right| \leq \frac{(x_2 - x_1) (S_3 - S_2)}{2}$$

on  $[x_1, u]$ . Hence, by Markov's inequality and the fact that  $h - f$  is a polynomial of degree  $n$  or less on  $[x_1, u]$ , we have, for  $\delta = u - x_1$ ,

$$(3.5) \quad \|h' - f'\|_{x_1, u} \leq \frac{(x_2 - x_1) (S_3 - S_2)}{\delta} n^2.$$

Now, if we combine (3.4), (3.5), and the fact that  $h'(x_1) = S_2$ , we have

$$S_2 - \frac{S_1(x_1 - x_0)}{\epsilon} n^2 \leq h'(x) - f'(x) \leq \frac{(x_2 - x_1) (S_3 - S_2)}{\delta} n^2,$$

and (3.3) follows. This completes the proof of Lemma 3.1. We now return to the

*Proof of Theorem 3.1.* Let  $n$  be given. Choose  $y_0, y_1, y_2, y_3$  so that

$$S_1 = \frac{x_1 - l}{4(x_1 - x_0)n^2}, \quad S_2 = 1 + S_1,$$

and

$$S_3 = S_2 + (u - x_1)/4(x_2 - x_1)n^2.$$

Then by (3.3), any  $f \in S_m^1(\Delta)$  satisfying (3.1) and (3.2) must satisfy

$$m^2 \geq 2n^2(1 + S_1).$$

That is,  $m > \sqrt{2(1 + S_1)}n > n$ . Hence, for this data no  $f \in S_n^1(\Delta)$  can satisfy (3.1) and (3.2). This establishes Theorem 3.1.

We note that Theorem 3.1 contains Theorem 3 in [6] as a special case.

**4. Variable Knots and Quadratic Splines.** In this section we show that for convex increasing data  $(x_i, y_i), i = 0, 1, \dots, N$ , it is always possible to construct a mesh  $\bar{\Delta}$  such that

$$\Delta = \{x_0, x_1, \dots, x_N\} \subset \bar{\Delta},$$

and  $\bar{\Delta}$  contains at most one additional knot between each pair  $x_{i-1}$  and  $x_i$  and such

that a quadratic spline  $f$  on  $\bar{\Delta}$  satisfies (2.2), (2.3), and (2.4). The proof will be constructive in that an algorithm for choosing the new knots will be given. In fact, we will present an algorithm for inserting new "data" so that (2.6) holds for this "expanded" set of data.

The following lemmas will be needed:

LEMMA 4.1. *Let the four data points  $(x_i, y_i)$ ,  $i = 0, 1, 2, 3$ , be convex and increasing, and let  $S_0$  be given so that  $0 < S_0 < S_1$ . Define*

$$(4.1) \quad \bar{x} = x_1 - 2(x_1 - x_0)(S_1 - S_0)/(S_2 - S_0),$$

$$(4.2) \quad \bar{y} = y_0 + S_0(\bar{x} - x_0)$$

and assume that  $x_0 < \bar{x} < x_1$  (in Lemma 4.2 we show that if  $m_2 \geq S_3$ , then  $x_0 < \bar{x} < x_1$ ).

Then

$$(4.3) \quad \frac{\bar{y} - y_0}{\bar{x} - x_0} = S_0;$$

$$(4.4) \quad \frac{y_1 - \bar{y}}{x_1 - \bar{x}} = \frac{S_2 - S_0}{2} + S_0 = \frac{S_2 + S_0}{2}.$$

(4.5) *The "expanded" data  $D = \{(x_0, y_0), (\bar{x}, \bar{y}), (x_1, y_1), (x_2, y_2), (x_3, y_3)\}$  are convex and increasing.*

(4.6) *There exists a set of  $\frac{1}{2}$ -admissible points for  $D$ .*

(4.7) *The  $m_i$  and  $M_i$  from the  $\frac{1}{2}$ -algorithm satisfy  $m_i < M_i$ ,  $0 \leq i \leq 2$ .*

(4.8) *There is a positive number  $e$  so that for any number  $m$  satisfying*

$$S_0 - e < m \leq S_0$$

*there is a  $\frac{1}{2}$ -admissible set  $T$  for the set  $D$  such that the piecewise linear function  $L$  associated with  $T$  (Definition 2.1) has first linear segment of slope  $m$ .*

*Proof.* Equations (4.3) and (4.4) are immediate consequences of (4.1) and (4.2). Property (4.5) follows from (4.3), (4.4) and the hypotheses. To prove (4.6) we observe that the  $\frac{1}{2}$ -algorithm for the "expanded" data set proceeds as follows: (Note that we have neither re-indexed the data nor renamed the slopes.)

$$\begin{aligned} m_0 &= 0 < S_0, & M_0 &= S_0; \quad \text{and} \\ m_1 &= S_0 < \frac{S_0 + S_2}{2}, & M_1 &= \min\left(\frac{S_2 + S_0}{2}, 2S_0\right). \end{aligned}$$

This gives two cases:

Case 1.

$$\begin{aligned} M_1 &= \frac{S_2 + S_0}{2}; \\ m_2 &= \frac{S_2 + S_0}{2} < S_2, & M_2 &= S_2; \quad \text{and} \\ m_3 &= S_2 < S_3, & M_3 &= \min\left(S_3, S_2 + \frac{S_2 - S_0}{2}\right). \end{aligned}$$

Case 2.

$$\begin{aligned}
 M_1 &= 2S_0; \\
 m_2 &= S_2 - S_0 < S_2, \quad M_2 = S_2; \quad \text{and} \\
 m_3 &= S_2 < S_3, \quad M_3 = \min(S_3, S_2 + S_0).
 \end{aligned}$$

Thus, (4.6) is established. Also, in both cases, we are guaranteed that  $m_i < M_i$  for  $i = 0, 1, 2, 3$  which gives (4.7). Thus, we may choose a  $\frac{1}{2}$ -admissible set  $\{t_1, t_2, t_3, t_4\}$  by choosing  $t_4$  to be any point lying between  $y_2 + m_3((x_3 - x_2)/2)$  and  $y_2 + M_3((x_3 - x_2)/2)$ . Furthermore, if we choose  $t_4 = y_2 + m_3((x_3 - x_2)/2)$ , then

$$(4.9) \quad t_4 = y_2 + S_2((x_3 - x_2)/2).$$

Now note that the line segment joining  $(x_0, y_0)$  and  $(\bar{x}, \bar{y})$  and the line segment joining  $(x_1, y_1)$  and  $(x_2, y_2)$  intersect at a point whose  $x$  coordinate is the midpoint of the segment  $[\bar{x}, x_1]$ . Furthermore, the  $y$  coordinate of the point where the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$  intersects the vertical line passing through  $(x_2 + x_3)/2$  is  $t_4$  as given by (4.9). Hence, for this choice of  $t_4$ , the resulting piecewise linear function  $L$  will have the slope of its first segment equal to  $S_0$ . Since any choice of  $t_4$  in the interval described above will give a different piecewise linear function  $L$  and since no such  $L$  can have the slope of its first segment greater than  $S_0$  and since  $M_3 > m_3$ , (4.8) is valid. This completes the proof of Lemma 4.1.

**LEMMA 4.2.** *Let the four data points  $(x_i, y_i)$ ,  $0 \leq i \leq 3$ , be convex and increasing and let  $S_0$  be given such that  $0 < S_0 < S_1$ . If in the  $\frac{1}{2}$ -algorithm  $m_2 \geq S_3$ , then  $\bar{x}$  as defined in (4.1) lies strictly between  $x_0$  and  $x_1$ .*

*Proof.* By definition  $m_2 = 2S_2 - M_1$  and  $M_1 = \min\{S_2, 2S_1 - m_0\} = \min\{S_2, 2S_1\}$ . If  $m_2 \geq S_3$ , then

$$2S_2 - \min\{S_2, 2S_1\} \geq S_3$$

from which it follows that  $M_1 = 2S_1$  and

$$2(S_2 - S_1) \geq S_3.$$

But  $S_3 > S_2$ . Hence,  $S_2 > 2S_1$ ; and therefore,  $S_2 - S_0 > 2(S_1 - S_0)$ . It follows that  $x_0 < \bar{x} < x_1$ , which establishes the lemma.

We now present the point insertion algorithm. Let  $(x_i, y_i)$ ,  $i = 0, 1, 2, \dots, N$ , be given convex and increasing data.

*Point insertion algorithm.* Apply the  $\frac{1}{2}$ -algorithm until

$$(4.10) \quad m_k \geq S_{k+1} \quad \text{for some } k$$

(note we are guaranteed that  $k \geq 2$ ). Consider the four points  $(x_{k-2}, y_{k-2})$ ,  $(x_{k-1}, y_{k-1})$ ,  $(x_k, y_k)$ ,  $(x_{k+1}, y_{k+1})$ . Apply Lemma 4.1 to these four points with

$$(4.11) \quad S_0 = (m_{k-2} + M_{k-2})/2.$$

Thus, we insert between  $(x_{k-2}, y_{k-2})$  and  $(x_{k-1}, y_{k-1})$  the point  $(\bar{x}, \bar{y})$  where  $\bar{x}$  and  $\bar{y}$  are from (4.1) and (4.2):

$$(4.12) \quad \bar{x} = x_{k-1} - 2(x_{k-1} - x_{k-2})(S_{k-1} - S_0)/(S_k - S_0)$$

and

$$(4.13) \quad \bar{y} = y_{k-2} + S_0(\bar{x} - x_{k-2}).$$

From Lemma 4.2,  $x_{k-2} < \bar{x} < x_{k-1}$  and  $y_{k-2} < \bar{y} < y_{k-1}$ .

Now renumber the points to include the point  $(\bar{x}, \bar{y})$ . Thus, the data now is  $(x_i, y_i)$  for  $i = 0, 1, \dots, N + 1$  with  $x_{k-1} = \bar{x}$  and  $y_{k-1} = \bar{y}$ . Now apply the  $\frac{1}{2}$ -algorithm to the "expanded" data until  $m_j \geq S_{j+1}$  and proceed as before. Note that it is only necessary to recalculate  $m_{k-2}$  and  $M_{k-2}$  and continue from this point since  $m_i$  and  $M_i$  for  $i = 0, 1, \dots, k - 3$  are the same as before.

**THEOREM 4.1.** *Given a convex, increasing set of data  $(x_i, y_i), i = 0, 1, \dots, N,$*

(4.14) *the point insertion algorithm will terminate after a finite number  $M \leq N$  of steps;*

(4.15) *at most one new point will be inserted between any two original data points;*

(4.16) *the "expanded" data set*

$$(x'_i, y'_i), \quad i = 0, 1, 2, \dots, N + M,$$

*thus obtained, will satisfy (2.6) and (2.7).*

*Hence, there exists  $f \in S_2^1(x'_0, x'_1, \dots, x'_{N+M})$  satisfying (2.2), (2.3), and (2.4).*

*Proof.* The  $\frac{1}{2}$ -algorithm will proceed until  $m_k \geq S_{k+1}$ . Let  $k$  be the smallest index for which this occurs. Then for all  $0 \leq j < k$  we have  $m_j < M_j$ . For suppose not. Let  $p$  be the smallest index for which  $m_p \geq M_p$ . Then  $p \geq 2$ . We note that

$$M_p - m_p = \min\{S_{p+1} - m_p, M_{p-1} - m_{p-1}\}.$$

Since  $m_p \geq M_p$  and  $M_{p-1} - m_{p-1} > 0$ , it follows that  $M_p = S_{p+1}$  and, therefore,  $m_p \geq S_{p+1}$  contradicting the definition of  $k$ . Therefore,  $m_j < M_j$  for all  $0 \leq j < k$ . Furthermore, since  $m_{k-2} < M_{k-2} \leq S_{k-1}$ , we have

$$m_{k-2} < S_0 < M_{k-2} \leq S_{k-1}.$$

Now insert the point  $(\bar{x}, \bar{y})$  between  $(x_{k-2}, y_{k-2})$  and  $(x_{k-1}, y_{k-1})$ , where  $\bar{x}$  and  $\bar{y}$  are given by (4.12) and (4.13). By (4.8) there is  $e > 0$  so that for any number  $m$  satisfying

$$S_0 - e < m \leq S_0$$

there is a  $\frac{1}{2}$ -admissible set for the five points  $(x_{k-2}, y_{k-2}), (\bar{x}, \bar{y}), (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1})$ , so that the piecewise linear function  $L$  associated with it has its first linear segment of slope  $m$ . We may assume that  $e$  is chosen small enough so that

$$m_{k-2} \leq S_0 - e.$$

If we re-index the data we get  $(x'_i, y'_i), i = 0, 1, \dots, N + 1$ , as above, it is now clear that if the  $\frac{1}{2}$ -algorithm is applied to this "expanded" data we will have

$$m_i < M_i \leq S_{i+1}$$

for  $i = 0, 1, 2, \dots, k + 2$ . Thus, a new point will not be inserted before  $(x'_k, y'_k)$ . That is, if another new point is to be inserted it will be after  $(x'_k, y'_k)$  (i.e., after  $(x_{k-1}, y_{k-1})$ ). This establishes (4.15) and, hence, (4.14). Assertion (4.16) is easily established by applying Lemmas 4.1 and 4.2. This completes the proof.

Finally, it can be shown that for  $\frac{1}{2}$ -admissible data, by adding a *single* knot in the interval  $[x_{N-1}, x_N]$ , a quadratic spline  $f$  can be constructed which satisfies (2.2), (2.3), (2.4) and  $f'(x_N) = S$  for any  $S > S_N$ . A similar result is true for  $[x_0, x_1]$  and  $S < S_0$ .

It follows that a quadratic spline exists which interpolates and "preserves the shape" of any finite set of data if at most one additional knot is added between any two original data points. Such a spline should be useful in computer aided geometric design and other applications of computer graphics.

**5. Numerical Examples and Remarks.** The point insertion algorithm described in Section 4 was applied to several sets of convex increasing data and in each case produced expanded data which were convex increasing and  $\frac{1}{2}$ -admissible. We report on three such cases because of their severe numerical properties. All calculations were done in double precision on an IBM 370-165. It should be noted that the data in examples 2 and 3 are so extreme that the point insertion algorithm failed to produce  $\frac{1}{2}$ -admissible data using only single precision.

TABLE 1

Example	Original Data			Points Added By Point Insertion Algorithm	
	$i$	$x_i$	$y_i$	$x$	$y$
1	0	0	0		
	1	2	2	1.902439024390243E + 00	9.512195121951219E - 01
	2	4	44		
	3	6	88		
2	0	0	0		
	1	2	2	1.902439024390243E + 00	9.512195121951219E - 01
	2	4	44	3.199999999999945E + 00	2.68999999999883E + 01
	3	6	88		
	4	8	132.1	7.999790784036821E + 00	1.320453920184109E + 02
	5	10	1132.1		
3	0	0	0		
	1	1	0.001	9.989994997498749E - 01	4.994997498749374E - 04
	2	2	1.001		
	3	3	2.002	2.999911763408285E + 00	2.001161741349135E + 00
	4	4	20.002		
	5	5	40.1	4.948832239073737E + 00	3.708109890736525E + 01
	6	6	140.1		
	7	7	282	6.917681846616026E + 00	2.318682420167180E + 02
	8	8	1400		
	9	9	2800	8.976580020835816E + 00	2.491816493812568E + 03
	10	10	28000		
	11	11	54000		
12	12	100000			



If the algorithm developed by McAllister, Passow, and Roulier [5] is applied to these sets of data, then the degree of the piecewise polynomial in  $S_n^1(\Delta)$  satisfying (2.2), (2.3), and (2.4) will be  $n = 21$  in example 1,  $n = 9560$  in example 2, and  $n = 1000$  in example 3. Indeed, any attempt to construct and evaluate the splines using the algorithm in [5] resulted in underflow and/or overflow when applied to examples 2 and 3.

On the other hand, if the algorithm in [5] is applied to the "expanded" data resulting from the point-insertion algorithm the shape-preserving piecewise quadratic spline can be constructed and evaluated in all three cases with no difficulty. The above-mentioned examples appear in Table 1.

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