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Unicity of Best Mean Approximation by Second Order Splines with Variable Knots

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Abstract. Let S_N^2 denote the nonlinear manifold of second order splines defined on $[0, 1]$ having at most N interior knots, counting multiplicities. We consider the question of unicity of best approximations to a function f by elements of S_N^2. Approximation relative to the $L_2[0, 1]$ norm is treated first, with the results then extended to the best L_1 and best one-sided L_1 approximation problems. The conclusions in each case are essentially the same, and can be summarized as follows: a sufficiently smooth function f satisfying $f'' > 0$ has a unique best approximant from S_N^2 provided either $\log f''$ is concave, or N is sufficiently large, $N \geqslant N_0(f)$; for any N, there is a smooth function f, with $f'' > 0$, having at least two best approximants. A principal tool in the analysis is the finite dimensional topological degree of a mapping.

1. **Introduction.** Let S_N^2 denote the nonlinear manifold of functions which are linear combinations of second order B-splines with at most N interior knots on $(0, 1)$ counting multiplicities. S_N^2 is the $L_2[0, 1]$ closure of the class of all piecewise linear continuous functions with at most $N + 1$ linear segments. In this article we prove some interesting and somewhat surprising approximation properties of S_N^2 in the space $L_2[0, 1]$. Three main theorems will be stated in this section with the proofs to follow in later sections. These results are announced in [1].

Theorem 1 describes a fairly large class of uniformly convex functions which have, for each positive integer N, unique best $L_2[0, 1]$ approximants from the (nonlinear) spline manifold S_N^2. Theorem 2 states that any sufficiently smooth uniformly convex function eventually, i.e. for all large N, has a unique best $L_2[0, 1]$ approximant from this manifold. This behavior will be called "eventual uniqueness". Theorem 3 indicates the sharpness of these two results.

We emphasize that S_N^2 is not a linear manifold. Hence, arguments regarding existence, uniqueness, and characterization for best approximants are nontrivial. Since it has been shown in [5] that, for every positive integer N, any continuous function has at least one best continuous $L_2[0, 1]$ approximant from S_N^2, we are only concerned with uniqueness and eventual uniqueness of best approximants in this paper.

THEOREM 1. *Let $f \in C^2[0, 1]$ with $f'' > 0$ on $[0, 1]$. Suppose that $\log f''$ is concave in $(0, 1)$. Then for every positive integer N, f has a unique best $L_2[0, 1]$ approximant from S_N^2.*

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THEOREM 2. *Let $f \in C^5[0, 1]$ with $f'' > 0$ on $[0, 1]$. Then there exists a positive integer N_0 such that for any integer $N > N_0$, f has a unique best $L_2[0, 1]$ approximant from S_N^2.*

THEOREM 3. *Let N be any positive integer. There exists a function $f \in C^\infty[0, 1]$ with $f'' > 0$ on $[0, 1]$, such that f has more than one best $L_2[0, 1]$ approximant from S_N^2.*

We mention that strictly convex $C^2[0, 1]$ functions always have unique best $L_\infty[0, 1]$ approximants from S_N^2 (cf. [7, p. 188]). Thus, Theorem 3 shows that there is a fundamental difference in the two norms regarding approximation from S_N^2, and in addition suggests that Theorem 1 is in some sense sharp. This seems to be quite interesting since the class of functions considered in Theorem 1 is not commonly associated with approximation theoretic questions. This work can be thought of as a first step in proving unicity theorems for (generalized) monosplines of minimal norm. The interested reader should consult the first two articles in [6] and the references therein. For a nonuniqueness result on monosplines with least L_2 norm, see Braess [4].

In establishing the above unicity and eventual unicity results, we will derive a very general method which can be used to yield analogous results in other settings. For example, in Section 7 we will consider the best approximation and best one-sided approximation problems in the L_1 norm.

2. Preliminaries and Notation. Let $\Sigma^N \subset R^N$ be the open simplex

$$\{\mathbf{t}^N = (t_1, \ldots, t_N): \ 0 < t_1 < \cdots < t_N < 1\}.$$

We will denote by $\mathbf{t}^N$ the N interior knots of a second order spline where $\mathbf{t}^N$ could be in the closure of Σ^N provided $t_{i+2} > t_i$ for $i = -1, \ldots, N$, where $t_{-1} = t_0 = 0$, $t_{N+1} = t_{N+2} = 1$. With this knot sequence, one can form the normalized B-splines

$$N_i(\mathbf{t}^N; \tau) = N_{i,2}(\mathbf{t}^N; \tau) = (t_{i+2} - t_i)[t_i, t_{i+1}, t_{i+2}]_s(s - \tau)_+$$

for $i = -1, \ldots, N$ (cf. [2]). Let $S_N^2 = S_N^2[0, 1]$ be the nonlinear manifold of spline functions $s(\cdot) = \sum_{i=-1}^N A_i N_i(\mathbf{t}^N; \cdot)$.

We are interested in the problem of uniqueness of best $L_2[0, 1]$ approximants from S_N^2. Let $f \in L_2[0, 1]$ and let $P_N(f)$ denote the collection of all best approximants to f from S_N^2 in the $L_2[0, 1]$ norm. Thus, $s \in P_N(f)$ if and only if $s \in S_N^2$ and

$$\int_0^1 |f - s|^2 = \inf\left\{\int_0^1 |f - g|^2 : g \in S_N^2\right\}.$$

It will be convenient at times to denote the dependence of the spline on its knot sequence. In this case we will write $s = s(\mathbf{t}^N)$ which will mean that

$$s(\cdot) = \sum_{i=-1}^N A_i N_i(\mathbf{t}^N; \cdot)$$

for some constants $A_{-1}, \ldots, A_N$.

3. Preliminary Lemmas. In this section we will state three lemmas which will be necessary in our proof of Theorems 1 and 2. The first lemma yields the variational equations which give a necessary condition satisfied by any best $L_2[0, 1]$ approximant from S_N^2.

LEMMA 1. *Let f be in $L_2[0, 1]$ and $s = s(\mathbf{t}^N)$ be in $P_N(f)$. Then the restriction of s to any of the subintervals (t_i, t_{i+1}), $i = 0, \ldots, N$, is a best linear L_2 approximant to f on this subinterval. That is,*

$$\int_{t_i}^{t_{i+1}} (f - s)(t) t^j \, dt = 0, \qquad j = 0, 1; i = 0, \ldots, N. \tag{3.1}$$

This lemma can be easily verified in a standard manner by differentiating the error with respect to both the linear parameters (the coefficients) and the nonlinear parameters (the knots).

The second lemma is a known result which gives some information on the placement of the knots of a best L_2 approximant.

LEMMA 2. *Let f be a continuous function on $[0, 1]$ such that $f \notin S_N^2$ and let $s(\mathbf{t}^N) \in P_N(f)$. Then $t_0 < t_1 < \cdots < t_{N+1}$ and $s'(t_i^+) \neq s'(t_i^-)$ for $i = 1, \ldots, N$. In particular, s is continuous on $[0, 1]$ and all the knots of s are active.*

The fact that all knots are active was apparently first observed by de Boor [3]. That s is continuous follows from Theorem 2.3 in [5].

Let l be the best linear $L_2[\alpha, \beta]$ approximant to f. Then it is easy to verify that

$$l(\alpha) = -6 \int_0^1 \left(\tau - \frac{2}{3}\right) f(\tau(\beta - \alpha) + \alpha) \, d\tau$$

and

$$l(\beta) = 6 \int_0^1 \left(\tau - \frac{1}{3}\right) f(\tau(\beta - \alpha) + \alpha) \, d\tau.$$

For any knot sequence $\mathbf{t}^N \in \Sigma^N$, let $\Delta t_{i-1} = t_i - t_{i-1}$, $1 \leqslant i \leqslant N + 1$, and consider the function $\mathbf{F} = \mathbf{F}(\mathbf{t}^N, f) = (F_1, \ldots, F_N)$, where

$$F_i = F_i(\mathbf{t}^N, f) = -\int_0^1 \left[\left(\tau - \frac{1}{3}\right) f(\tau \Delta t_{i-1} + t_{i-1}) + \left(\tau - \frac{2}{3}\right) f(\tau \Delta t_i + t_i)\right] d\tau. \tag{3.2}$$

Note that $\mathbf{F}$ is defined and continuous on the closure of Σ^N. From the above two lemmas, we know that if $s = s(\mathbf{t}^N)$ is a best $L_2[0, 1]$ approximant from S_N^2 to a continuous function f, then $\mathbf{F}(\mathbf{t}^N, f) = \mathbf{0}$ where $\mathbf{0} = (0, \ldots, 0)$. Also, for any knot sequence $\mathbf{t}^N$, the Jacobian matrix $J(\mathbf{F}(\mathbf{t}^N)) \equiv (\alpha_{i,j})(f)$ of $\mathbf{F}$ defined above as a function of $t_1, \ldots, t_N$ is a tridiagonal matrix; and if f is in $C^2[0, 1]$, the only nonzero entries (obtained by differentiating (3.2) and then integrating by parts) are

$$\alpha_{i,i-1}(f) = \frac{\partial F_i}{\partial t_{i-1}} = -\frac{\Delta t_{i-1}}{3} \int_0^1 \tau(1 - \tau)^2 f''(\tau \Delta t_{i-1} + t_{i-1}) d\tau, \qquad 2 \leqslant i \leqslant N, \tag{3.3}$$

$$\begin{aligned} \alpha_{i,i}(f) = \frac{\partial F_i}{\partial t_i} &= -\frac{\Delta t_i}{6} \int_0^1 (2\tau - 1)(1 - \tau)^2 f''(\tau \Delta t_i + t_i) \, d\tau \\ &\quad + \frac{\Delta t_{i-1}}{6} \int_0^1 (2\tau - 1)\tau^2 f''(\tau \Delta t_{i-1} + t_{i-1}) d\tau, \qquad 1 \leqslant i \leqslant N, \end{aligned} \tag{3.4}$$

and

$$\alpha_{i,i+1}(f) = \frac{\partial F_i}{\partial t_{i+1}} = -\frac{\Delta t_i}{3}\int_0^1 \tau^2(1-\tau)f''(\tau\Delta t_i + t_i)\,d\tau, \qquad 1 \leqslant i \leqslant N-1. \tag{3.5}$$

We have the following result.

LEMMA 3. *Let f be in $C^2[0, 1]$ having the properties that $f'' > 0$ and $\log(f'')$ is concave (i.e. f'''/f'' is nonincreasing a.e.). If $\mathbf{F}(\mathbf{t}^N) = \mathbf{0}$, then the determinant of $J(\mathbf{F}(\mathbf{t}^N))$ is positive.*

Here, and throughout, det A denotes the determinant of a matrix A. To prove this lemma, we first verify that for each i, $i = 2, \ldots, N-1$,

$$\begin{aligned}\sum_{j=1}^{N} \alpha_{i,j}(f) = &-\frac{(\Delta t_i)^2}{6}\int_0^1 \tau(1-\tau)^2 f'''(\tau\Delta t_i + t_i)\,d\tau \\ &+ \frac{(\Delta t_{i-1})^2}{6}\int_0^1 \tau^2(1-\tau)f'''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau.\end{aligned} \tag{3.6}$$

Indeed, this follows from integrating (3.2) by parts twice, yielding

$$\begin{aligned}F_i(\mathbf{t}^N, f) = &-\frac{(\Delta t_i)^2}{6}\int_0^1 \tau(1-\tau)^2 f''(\tau\Delta t_i + t_i)\,d\tau \\ &+ \frac{(\Delta t_{i-1})^2}{6}\int_0^1 \tau^2(1-\tau)f''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau.\end{aligned} \tag{3.7}$$

If $\mathbf{t}^N$ is a solution to $\mathbf{F}(\mathbf{t}^N, f) = \mathbf{0}$, then by (3.6) and (3.7), we have, for $i = 2, \ldots, N-1$,

$$\begin{aligned}\sum_{j=1}^{N} \alpha_{i,j}(f) = -C\Bigg[&\int_0^1 \tau^2(1-\tau)f''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau \\ &\cdot\int_0^1 \tau(1-\tau)^2 f'''(\tau\Delta t_i + t_i)\,d\tau \\ &-\int_0^1 \tau(1-\tau)^2 f''(\tau\Delta t_i + t_i)\,d\tau \\ &\cdot\int_0^1 \tau^2(1-\tau)f'''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau\Bigg],\end{aligned} \tag{3.8}$$

where

$$C = \frac{(\Delta t_{i-1})^2}{6}\Big/\int_0^1 \tau(1-\tau)^2 f''(\tau\Delta t_i + t_i)\,d\tau.$$

Multiplying and dividing the right side of (3.8) by

$$\left(\int_0^1 \tau^2(1-\tau)f''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau\right)\left(\int_0^1 \tau(1-\tau)^2 f''(\tau\Delta t_i + t_i)\,d\tau\right)$$

yields

$$\sum_{j=1}^{N} \alpha_{ij}(f) = P[A_{i-1} - A_i], \tag{3.9}$$

where $P > 0$,

$$A_i = \int_0^1 \tau(1-\tau)^2 f'''(\tau\Delta t_i + t_i)\,d\tau \Big/ \int_0^1 \tau(1-\tau)^2 f''(\tau\Delta t_i + t_i)\,d\tau$$

and

$$A_{i-1} = \int_0^1 \tau^2(1-\tau) f'''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau \Big/ \int_0^1 \tau^2(1-\tau) f''(\tau\Delta t_{i-1} + t_{i-1})\,d\tau.$$

Noting that $f''' = (f'''/f'')f''$, we see that A_i is a weighted average of f'''/f'' over the interval (t_i, t_{i+1}). Since f'''/f'' is nonincreasing, we conclude that $\Sigma_{j=1}^N \alpha_{i,j}(f) \geqslant 0$ for $i = 2, \ldots, N-1$.

For $i = 1$ and N we can conclude that $\Sigma_{j=1}^N \alpha_{ij}(f) > 0$ by simply noting that $\alpha_{1,0}$ and $\alpha_{N,N+1}$ are well defined by the right side of (3.3) and (3.5), respectively, and arguing as above. Since the off diagonal elements $\alpha_{i,i-1}(f)$ and $\alpha_{i,i+1}(f)$ are negative by using (3.3) and (3.5), we see that the matrix $J(\mathbf{F}(\mathbf{t}^N))$ is diagonally dominant with strict inequality in the first and last rows. By a standard modification of Gershgorin's theorem, it follows that all the eigenvalues of $J(\mathbf{F}(\mathbf{t}^N))$ are in the open right half of the complex plane. Furthermore, since the matrix has only real entries, the complex eigenvalues come in conjugate pairs, so that the product of all the eigenvalues of $J(\mathbf{F}(\mathbf{t}^N))$ is positive, and Lemma 3 is proved.

4. Proof of the Uniqueness Theorem. We are now ready to prove Theorem 1. Set $g(x) = x^2$. Hence, if $s = s(\mathbf{t}^N)$ is a best $L_2[0, 1]$ approximant to g from S_N^2, then the only solution to $\mathbf{F}(\mathbf{t}^N, g) = \mathbf{0}$ satisfies $t_i = i/(N+1)$, $i = 1, \ldots, N$, as can easily be seen from (3.7). That is, $g(x) = x^2$ has a unique best $L_2[0, 1]$ approximant from S_N^2, and the knots of this approximant are equally spaced on $[0, 1]$.

Let f be as in Theorem 1. We will show that $\mathbf{F}(\mathbf{t}^N, f) = \mathbf{0}$ has exactly one solution among all possible knot sequences $\mathbf{t}^N$. Recall that the topological degree of the smooth mapping $\mathbf{G}$ from a bounded open set D in $\mathbf{R}^N$ into $\mathbf{R}^N$, where $\mathbf{G}$ is continuous on the closure of D, is given (cf. [8, p. 69]) by

$$\deg(\mathbf{p}, \mathbf{G}, D) = \sum_{\mathbf{G}(\mathbf{x})=\mathbf{p}} \operatorname{sign} \det J(\mathbf{G}(\mathbf{x})), \tag{4.1}$$

where the sum is taken over all solutions $\mathbf{x} \in D$ of $\mathbf{G}(\mathbf{x}) = \mathbf{p}$, as long as the Jacobian does not vanish at $\mathbf{x}$ and $\mathbf{p} \notin \mathbf{G}(\partial D)$. It is known that the degree is invariant under homotopy provided that the functions in the homotopy do not introduce solutions on the boundary of D. Again, let $g(x) = x^2$ and set

$$\mathbf{F}^\lambda(\cdot) \equiv \mathbf{F}(\cdot, (1-\lambda)g + \lambda f).$$

Clearly, $\lambda \longrightarrow \mathbf{F}^\lambda$ is a homotopy. Then by (3.7) it is easy to see that $\mathbf{F}^\lambda$ does not vanish on the boundary of Σ^N. Hence, it follows that

$$\deg(\mathbf{0}, \mathbf{F}^\lambda, \Sigma^N) = \deg(\mathbf{0}, \mathbf{F}^0, \Sigma^N) \tag{4.2}$$

for each λ, $0 \leqslant \lambda \leqslant 1$. Therefore, from Lemma 3, by using (4.2), we conclude that the number of solutions of $\mathbf{F}^1(\mathbf{t}^N) = \mathbf{0}$ is $\deg(\mathbf{0}, \mathbf{F}^0, \Sigma^N)$. From the fact that the determinant of $J(\mathbf{F}^0(\mathbf{t}^N))$ is positive and the fact that $\mathbf{F}(\mathbf{t}^N, g) = \mathbf{0}$ has only one solution, we conclude that $\deg(\mathbf{0}, \mathbf{F}^0, \Sigma^N) = 1$. This completes the proof of the theorem.

5. Proof of the Eventual Uniqueness Theorem. Let f satisfy the hypotheses of Theorem 2. By arguing as in Section 4, we obtain

$$\deg(\mathbf{0}, \mathbf{F}, \Sigma^N) = 1, \tag{5.1}$$

where $\mathbf{F}(\mathbf{t}^N) \equiv \mathbf{F}(\mathbf{t}^N, f)$ is defined by (3.2). To prove Theorem 2, we must show that

$$\det J(\mathbf{F}(\mathbf{t}^N)) > 0, \tag{5.2}$$

whenever $\mathbf{t}^N$ solves $\mathbf{F}(\mathbf{t}^N) = \mathbf{0}$ and N is sufficiently large. The fact that $\mathbf{t}^N$ is then the unique solution follows as in Section 4.

The key to proving (5.2) is the following algebraic result.

PROPOSITION 1. *Let $A = (a_{i,j})$ be a tridiagonal $N \times N$ real matrix with positive diagonal entries. Then if*

$$a_{n,n-1}a_{n-1,n} \leqslant a_{n,n}a_{n-1,n-1}(1 + \pi^2/4N^2)/4 \tag{5.3}$$

for $n = 2, \ldots, N$, it follows that

$$\det A > 0. \tag{5.4}$$

Proof. Let $A' = \operatorname{diag}(2/a_{ii})A$, so that A' has all 2's on the main diagonal and $\det A' > 0$ if and only if $\det A > 0$. For $n = 1, 2, \ldots, N$, let u_n be the determinant of the upper left $n \times n$ submatrix of A'. Then an expansion of u_n about the nth column yields

$$u_n = 2u_{n-1} - \left(\frac{2a_{n-1,n}}{a_{n-1,n-1}}\right)\left(\frac{2a_{n,n-1}}{a_{n,n}}\right)u_{n-2}, \tag{5.5}$$

$n = 2, \ldots, N$, where we define $u_0 = 1$ and $u_1 = 2$. This can be written as

$$u_n - 2u_{n-1} + u_{n-2} \equiv \Delta^2 u_{n-2} = c_n u_{n-2}, \tag{5.6}$$

where

$$c_n = 1 - \frac{4a_{n-1,n}a_{n,n-1}}{a_{n-1,n-1}a_{n,n}} \geqslant -\frac{\pi^2}{4N^2},$$

by (5.3).

Before showing that each $u_n > 0$, $n = 1, \ldots, N$, and hence that $\det A > 0$, we motivate a key equation ((5.8) below) with the following observation. Suppose that u and v solve the problems

$$u''(x) = f(x)u(x), \quad x \geqslant 0, \qquad u(0) = u'(0) = 1,$$

$$v''(x) = -\omega^2 v(x), \quad x \geqslant 0, \qquad v(0) = v'(0) = 1.$$

Then $w \equiv u - v$ satisfies

$$w''(x) = -\omega^2 w(x) + (f(x) + \omega^2)u(x), \quad w(0) = w'(0) = 0,$$

and one obtains

$$w(x) = \int_0^x \frac{\sin \omega(x - \xi)}{\omega} (f(\xi) + \omega^2)u(\xi)d\xi, \qquad x \geqslant 0.$$

We remark that the Green's function $\sin \omega(x - \xi)/\omega$ can be written as $a(x - \xi)$ where a solves the problem $a''(x) = -\omega^2 a(x)$, $a(0) = 0$, $a'(0) = 1$.

We now return to the proof of Proposition 1. Let $\{v_n\}$, $\{a_n\}$, $n = 0, \ldots, N$, satisfy

$$\Delta^2 v_{n-2} = v_n - 2v_{n-1} + v_{n-2} = -\omega^2 v_{n-2}, \quad n \geqslant 2, v_0 = 1, v_1 = 2,$$

and

$$\Delta^2 a_{n-2} = -\omega^2 a_{n-2}, \qquad n \geqslant 2, a_0 = 0, a_1 = 1.$$

Now let $w_n = u_n - v_n$, so that

$$\Delta^2 w_{n-2} = -\omega^2 w_{n-2} + (c_n + \omega^2)u_{n-2}, \qquad n \geqslant 2, w_0 = w_1 = 0. \tag{5.7}$$

We claim that the sequence defined by

$$x_n = \sum_{k=0}^{n-1} a_{n-1-k}(c_{k+2} + \omega^2)u_k, \qquad n \geqslant 1, x_0 = 0, \tag{5.8}$$

equals $\{w_n\}$. Indeed, $x_1 = 0$ since $a_0 = 0$; and we have only to prove that (5.7) holds. A direct calculation gives, for $n \geqslant 2$,

$$\Delta^2 x_{n-2} = x_n - 2x_{n-1} + x_{n-2} = \sum_{k=0}^{n-1} a_{n-1-k}(c_{k+2} + \omega^2)u_k$$

$$- 2\sum_{k=0}^{n-2} a_{n-2-k}(c_{k+2} + \omega^2)u_k + \sum_{k=0}^{n-3} a_{n-3-k}(c_{k+2} + \omega^2)u_k$$

$$= (c_n + \omega^2)u_{n-2} + \sum_{k=2}^{n-1} (c_{n-k+1} + \omega^2)u_{n-k-1}\, \Delta^2 a_k$$

$$= (c_n + \omega^2)u_{n-2} + \sum_{j=0}^{n-3} (c_{j+2} + \omega^2)u_j(-\omega^2 a_{n-3-j})$$

$$= -\omega^2 x_{n-2} + (c_n + \omega^2)u_{n-2},$$

and so the claim that $w_n = x_n$ is proved.

Solving for $\{v_n\}$, $\{a_n\}$ yields

$$v_n = \mathrm{Re}(1 + i\omega)^n + \mathrm{Im}(1 + i\omega)^n/\omega, \quad \text{and} \quad a_n = \mathrm{Im}(1 + i\omega)^n/\omega.$$

Consequently, if $N\omega \leqslant \pi/2$, we have $v_n > 0$ for $n = 0, \ldots, N$ and $a_n > 0$ for $n = 1, \ldots, N$. Finally, using (5.8) for w_n, with ω replaced by $\pi/2N$ so that $c_{k+2} + \omega^2$

$\geqslant 0$, one concludes that $w_n < 0$ is impossible; and hence,

$$u_n = w_n + v_n > 0, \qquad n = 1, \ldots, N.$$

This completes the proof of Proposition 1.

Before we can use this proposition, we will need information about the rate at which the Δt_i's tend to zero as N increases. The next lemma shows that a mesh, $\mathbf{t}^N$, which solves $\mathbf{F}(\mathbf{t}^N, f) = \mathbf{0}$ is in fact *quasi-uniform.*

It will be convenient at this point to introduce some notation which will facilitate the application of some of the results in this section to Section 7. For $\mathbf{t}^N \in \Sigma^N$, let

$$h_i = \Delta t_i = t_{i+1} - t_i, \quad i = 0, \ldots, N, \quad \text{and} \quad \Delta = \max h_i, \quad \delta = \min h_i.$$

We rewrite Eq. (3.7) in the form

$$F_i(\mathbf{t}^N) = h_{i-1}^2 \int_0^1 w(\tau) g(t_i - \tau h_{i-1}) d\tau - h_i^2 \int_0^1 w(\tau) g(t_i + \tau h_i) d\tau, \tag{5.9}$$

where $w(\tau) = \tau(1 - \tau)^2/6$ and $g = f''$.

LEMMA 4. *There is a constant* $M > 0$, *depending on f but not on N, such that if* $F(\mathbf{t}^N) = \mathbf{0}$, $\Delta/\delta \leqslant M$.

Proof. Let $0 < m = \min g(t)$ and $K = \max |g'(t)|, 0 \leqslant t \leqslant 1$. From (5.9) with $F_i = 0$, we obtain

$$h_i^2 g(\eta_i) = h_{i-1}^2\, g(\eta_{i-1}) = h_{i-1}^2 (g(\eta_i) + \theta_i(\eta_{i-1} - \eta_i)),$$

where $t_{i-1} < \eta_{i-1} < t_i < \eta_i < t_{i+1}$ and $|\theta_i| \leqslant K$. Let $\Delta = h_J$ and $\delta = h_I$, where we assume without loss of generality that $J > I$. Then

$$(\Delta/\delta)^2 = \prod_{i=I+1}^{J} \left(\frac{h_i}{h_{i-1}}\right)^2 = \prod_{i=I+1}^{J} \left[1 + \frac{\theta_i}{g(\eta_i)} (\eta_{i-1} - \eta_i)\right]$$

$$\leqslant \exp\left(\sum_{i=1}^{N} \left|\frac{\theta_i}{g(\eta_i)} (h_i + h_{i-1})\right|\right) \leqslant \exp(2K/m) \equiv M^2.$$

COROLLARY. $\Delta \leqslant M/N$ and $\delta \geqslant 1/M(N + 1)$.

Proof. Use $(N + 1)\delta \leqslant 1$ and $(N + 1)\Delta \geqslant 1$.

In the following, we will use the expression "$O(\Delta^p)$", where p is a positive integer. This will mean that there are positive numbers K and Δ_0, which will depend on f and w but not on N, such that

$$|O(\Delta^p)| \leqslant K\Delta^p \quad \text{if } \Delta \leqslant \Delta_0.$$

The following lemma, in conjunction with Proposition 1 and Lemma 4, will complete the proof of Theorem 2.

LEMMA 5. *Let* $\mathbf{t}^N$ *solve the equation* $\mathbf{F}(\mathbf{t}^N) = \mathbf{0}$, *and let* $J(\mathbf{F}(\mathbf{t}^N)) = (\alpha_{i,j})$. *Then for N sufficiently large,*

$$\alpha_{i,i} > 0, \quad i = 1, \ldots, N, \quad \textit{and}$$

$$\alpha_{i,i+1}\alpha_{i+1,i} = \alpha_{i,i}\alpha_{i+1,i+1}\,(1 + O(\Delta^3))/4, \qquad i = 1, \ldots, N-1.$$

Proof. Let g_i denote $g(t_i)$, $g_i' = g'(t_i)$, etc. Set $F_i = 0$ in (5.9), and expand g about t_i, obtaining

$$h_{i-1}^2 \int_0^1 w(\tau)(g_i - \tau h_{i-1} g_i' + \tau^2 h_{i-1}^2 g_i''/2 + O(\Delta^3))\,d\tau$$

$$= h_i^2 \int_0^1 w(\tau)(g_i + \tau h_i g_i' + \tau^2 h_i^2 g_i''/2 + O(\Delta^3))\,d\tau.$$

Let

$$A = \int_0^1 w(\tau)\,d\tau, \quad B = \int_0^1 \tau w(\tau)\,d\tau \quad \text{and} \quad C = \int_0^1 \tau^2 w(\tau)\,d\tau/2.$$

This yields the expressions

$$\text{(5.10a)} \qquad h_{i-1}^2 - h_i^2 = \frac{Bg_i'}{Ag_i}(h_i^3 + h_{i-1}^3) + O(\Delta^5),$$

$$\text{(5.10b)} \qquad h_{i-1} - h_i = \frac{Bg_i'}{Ag_i}(h_i^2 - h_i h_{i-1} + h_{i-1}^2) + O(\Delta^4).$$

If one differentiates (5.9) and then expands g about t_i, one obtains

$$\frac{\partial F_i}{\partial t_{i-1}} = h_{i-1}(-2Ag_i + 3Bh_{i-1}g_i' - 4Ch_{i-1}^2 g_i'' + O(\Delta^3)),$$

$$\frac{\partial F_i}{\partial t_i} = (h_{i-1} + h_i)2Ag_i + (h_{i-1}^2 - h_i^2)(A - 3B)g_i'$$

$$+ (h_{i-1}^3 + h_i^3)(-B + 4C)g_i'' + O(\Delta^4),$$

$$\frac{\partial F_i}{\partial t_{i+1}} = h_i(-2Ag_i - 3Bh_i g_i' - 4Ch_i^2 g_i'' + O(\Delta^3)).$$

A straightforward calculation gives

$$\frac{\alpha_{i,i+1}\alpha_{i+1,i}}{4h_i^2A^2g_ig_{i+1}} = 1 + h_i\left(\frac{3B}{2A}\right)\left(\frac{g_i'}{g_i} - \frac{g_{i+1}'}{g_{i+1}}\right)$$

$$\text{(5.11)} \qquad + h_i^2\left[\frac{4C}{2A}\left(\frac{g_{i+1}''}{g_{i+1}} + \frac{g_i''}{g_i}\right) - \frac{9B^2}{4A^2}\frac{g_i'}{g_i}\frac{g_{i+1}'}{g_{i+1}}\right] + O(\Delta^3)$$

$$= 1 + h_i^2\left[\frac{g_i''}{g_i}\left(\frac{-3B + 8C}{2A}\right) + \left(\frac{g_i'}{g_i}\right)^2\left(\frac{6AB - 9B^2}{4A^2}\right)\right] + O(\Delta^3).$$

Similarly, letting $k_i = (h_i + h_{i-1})/2h_i$, and $k_{i+1} = (h_{i+1} + h_i)/2h_i$, one computes

$$\frac{\alpha_{i,i}\alpha_{i+1,i+1}}{16h_i^2A^2g_ig_{i+1}} = k_ik_{i+1}$$

$$(5.12) \quad + k_i\left[\frac{(h_i^2 - h_{i+1}^2)}{2h_i}\left(\frac{A-3B}{2A}\right)\frac{g'_{i+1}}{g_{i+1}} + \frac{(h_i^3 + h_{i+1}^3)}{2h_i}\left(\frac{-B+4C}{2A}\right)\frac{g''_i}{g_i}\right]$$

$$+ k_{i+1}\left[\frac{(h_{i-1}^2 - h_i^2)}{2h_i}\left(\frac{A-3B}{2A}\right)\frac{g'_i}{g_i} + \frac{(h_{i-1}^3 + h_i^3)}{2h_i}\left(\frac{-B+4C}{2A}\right)\frac{g''_i}{g_i}\right] + O(\Delta^3),$$

where we have used (5.10a) to combine the term involving $(h_{i-1}^2 - h_i^2)(h_i^2 - h_{i+1}^2)$ with the higher order terms. We now use the Eqs. (5.10) to obtain

$$k_ik_{i+1} = 1 + h_i^2\left[-\frac{1B}{2A}\left(\frac{g'_i}{g_i}\right)' + \frac{3}{4}\left(\frac{B}{A}\frac{g'_i}{g_i}\right)^2\right] + O(\Delta^3).$$

Finally, if we use (5.10a) to substitute for the term $h_i^2 - h_{i+1}^2$ and $h_{i-1}^2 - h_i^2$ in (5.12), we arrive at

$$(5.13) \quad \frac{\alpha_{i,i}\alpha_{i+1,i+1}}{16h_i^2A^2g_ig_{i+1}} = 1 + h_i^2\left[\frac{g''_i}{g_i}\left(\frac{-3B+8C}{2A}\right) + \left(\frac{g'_i}{g_i}\right)^2\left(\frac{6AB-9B^2}{4A^2}\right)\right] + O(\Delta^3).$$

A comparison of (5.13) with (5.11) shows that Lemma 5 is correct, and hence completes the proof of Theorem 2.

6. Proof of the Nonuniqueness Theorem. We now prove Theorem 3. Let N be a positive integer. We first define a function $f_l = f_{l,N}$, $l > 0$, depending on whether N is even or odd, as follows: Let $N = 2k$ or $2k - 1$. Choose an angle θ with $0 < \theta < \pi/(4(k+1))$. If $N = 1$ (or $k = 1$), then the choice of θ is not necessary.

(i) For $N = 2k$, let f_l be a continuous piecewise linear function on $[0, kl + 1]$ with $f_l(0) = 0$ and with slopes equal to $\tan j\theta$ on $((j-1)l, jl)$, $j = 1, \ldots, k$, and 1 on $(kl, kl + 1]$.

(ii) For $N = 2k - 1$, let f_l be a continuous piecewise linear function on $[0, kl + 1]$ with $f_l(0) = 0$ and with slopes equal to $\tan(j-1)\theta$ on $((j-1)l, jl)$, $j = 1, \ldots, k$, and 1 on $(kl, kl + 1]$.

We next extend f_l to be an even function on $[-kl - 1, kl + 1]$, i.e. $f_l(-x) = f_l(x)$, $x \in [0, kl + 1]$. Then f_l is convex and belongs to $S_{N+1}^2[-kl - 1, kl + 1]$.

Let $s_l^* \in S_N^2[-kl - 1, kl + 1]$ be defined on $(-kl - 1, kl)$ as the restriction of f_l. Hence, s_l^* is continuous at kl and actually linear on $[(k-1)l, kl + 1]$. Let $\epsilon \equiv \|f_l - s_l^*\|_2 > 0$; note that ϵ is independent of l. Here, and throughout this proof, the L_2 norms are taken on the interval $[-kl - 1, kl + 1]$. Suppose that f_l has a unique best L_2 approximant s_l from $S_N^2[-kl - 1, kl + 1]$. Then s_l must be an even function, and for large l the knots $x_i = x_i(l)$, $i = 1, \ldots, N$, of s_l must interlace the knots of f_l in the following manner: $-kl < x_1 < -(k-1)l < x_2 < \cdots < x_{N-1} < (k-1)l < x_N < kl$, with the exception that if N is odd ($N = 2k - 1$), then the middle knot x_k

$= 0$ is the only knot that lies in the interval $(-l, l)$. It can also be shown that as $l \longrightarrow \infty$, then $kl - x_N = kl - x_N(l) \longrightarrow \infty$, since the error must remain below ϵ. Hence, $s_l(t) - f_l(t) \longrightarrow 0$ for $t \in (x_N(l), kl)$, and we have

$$\liminf_{l \to \infty} \|f_l - s_l\|_2 \geqslant 2\epsilon,$$

where one ϵ is obtained from the error estimate on each of the intervals $[-kl - 1, -kl]$ and $[kl, kl + 1]$. Thus, there is a positive l_0, such that for each $l \geqslant l_0$,

$$\|f_l - s_l^*\|_2 < \|f_l - s_l\|_2,$$

contradicting that s_l is the best L_2 approximant of f_l from $S_N^2[-kl - 1, kl + 1]$. In particular, f_{l_0} has more than one best L_2 approximant from $S_N^2[-kl_0 - 1, kl_0 + 1]$. By a standard smoothing technique, one can smooth the "corners" of f_{l_0}, and in fact, make f_{l_0} strictly convex, without affecting the best approximants too much. Finally, by a simple translation, one can change the interval from $[-kl_0 - 1, kl_0 + 1]$ to $[0, 1]$.

7. Application to L_1 Approximation. We now turn our attention to best $L_1\,[0, 1]$ approximation of convex functions from S_N^2. In fact, we will show that Theorems 1, 2 and 3 hold when L_2 is replaced by L_1. Furthermore, essentially the same results hold for a one-sided best L_1 approximation problem as indicated below.

We first note that Lemma 1 holds in the L_1 case if $(f - s)$ is replaced by $\operatorname{sgn}(f - s)$ and Lemma 2 is also true in L_1 [9]. If f is a convex function on the interval $[\alpha, \beta]$ then the best linear $L_1\,[\alpha, \beta]$ approximant is obtained by interpolating f at $\alpha + (\beta - \alpha)/4$ and $\alpha + 3(\beta - \alpha)/4$. In analogy with Eq. (3.2) we define the functional

$$\begin{aligned} F_i = F_i(\mathbf{t}^N, f) &= f(t_i + h_i/4) - [f(t_i + 3h_i/4) - f(t_i + h_i/4)]/2 \\ &\quad - \{f(t_{i-1} + h_{i-1}/4) + 3[f(t_{i-1} + 3h_{i-1}/4) - f(t_{i-1} + h_{i-1}/4)]/2\}. \end{aligned} \tag{7.1}$$

Setting $\mathbf{F} = \mathbf{F}(\mathbf{t}^N, f) = (F_1, \ldots, F_N)$, we see that if f is convex and $s(\mathbf{t}^N; \cdot)$ is a best $L_1\,[0, 1]$ approximant to f, then $\mathbf{F}(\mathbf{t}^N, f) = \mathbf{0}$.

One can now compute $J(\mathbf{F}(\mathbf{t}^N))$ as in Section 3, yielding for instance

$$\alpha_{i,i-1} = -h_{i-1}(3/8)\int_{1/4}^{3/4} f''(t_{i-1} + \tau h_{i-1})d\tau, \qquad 2 \leqslant i \leqslant N, \tag{7.2}$$

$$\alpha_{i,i+1} = -h_i(3/8)\int_{1/4}^{3/4} f''(t_i + \tau h_i)d\tau, \qquad 1 \leqslant i \leqslant N - 1. \tag{7.3}$$

Furthermore, (7.1) can be rewritten in the form

$$\begin{aligned} F_i(\mathbf{t}^N) &= h_{i-1}^2 \int_0^1 \hat{w}(\tau) f''(t_i - \tau h_{i-1})\, d\tau \\ &\quad - h_i^2 \int_0^1 \hat{w}(\tau) f''(t_i + \tau h_{i-1})\, d\tau, \end{aligned} \tag{7.4}$$

where $\hat{w}(\tau) = \tau - 3/2(\tau - 1/4)_+ + ½(\tau - ¾)_+$ is a nonnegative weight function on $[0, 1]$.

We are now in a position to state

THEOREM 4. *Let* $f \in C^2[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$. *If* $\log f''$ *is concave on* $(0, 1)$, *then there exists a unique* $s^* \in S_N^2$ *which is a best* $L_1[0, 1]$ *approximant to f.*

THEOREM 5. *Let* $f \in C^5[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$. *There exists a positive integer* N_0 *such that for every* $N \geq N_0$, *f has a unique best* $L_1[0, 1]$ *approximant from* S_N^2.

THEOREM 6. *For any integer N there exists a function* $f \in C^\infty[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$ *which does not have a unique best* $L_1[0, 1]$ *approximant from* S_N^2.

The proofs of Theorems 4, 5, and 6 closely parallel those of Theorems 1, 2, and 3, respectively.

We next consider the following one-sided best $L_1[0, 1]$ approximation problem from S_N^2. This problem can be treated as a problem of interpolation by functions from S_N^2 at the (variable) knots with minimum error. Let f be a uniformly convex function on $[0, 1]$. The problem is to study the uniqueness and eventual uniqueness of an $s^*(\cdot) = s^*(\mathbf{t}^N, \cdot)$ from S_N^2, with knot sequence $\mathbf{t}^N = (t_1^*, \ldots, t_N^*) \in \Sigma^N$, such that

$$s^*(t_i^*) = f(t_i^*), \qquad i = 0, \ldots, N+1, \quad \text{and} \tag{7.5}$$

$$\|f - s^*\|_1 = \inf\{\|f - s\|_1 : s(\cdot) = s(\mathbf{t}^N; \cdot) \in S_N^2,\ s(t_i) = f(t_i),\ 0 \leq i \leq N+1\}. \tag{7.6}$$

We have the following results.

THEOREM 7. *Let* $f \in C^2[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$. *For* $N = 1$, *there exists a unique* $s^* \in S_1^2$ *satisfying* (7.5) *and* (7.6). *For* $N \geq 2$ *if, in addition,* $\log f''$ *is concave on* $(0, 1)$, *then there exists a unique* $s^* \in S_N^2$ *satisfying* (7.5) *and* (7.6).

THEOREM 8. *Let* $f \in C^5[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$. *There exists a positive integer* N_0 *such that for every* $N \geq N_0$, *there is a unique* $s^* \in S_N^2$ *satisfying* (7.5) *and* (7.6).

THEOREM 9. *Let* $N \geq 2$. *There exists a function* $f \in C^\infty[0, 1]$ *with* $f'' > 0$ *on* $[0, 1]$, *such that the function* $s^* \in S_N^2$ *that satisfies* (7.5) *and* (7.6) *is not unique.*

In proving Theorems 7 and 8, we again derive the analogous quantities F_i, namely,

$$\begin{aligned} F_i &= f(t_{i+1}) - f(t_{i-1}) - f'(t_i)(t_{i+1} - t_{i-1}) \\ &= h_i^2 \int_0^1 \widetilde{w}(t) f''(t_i + th_i)\, dt - h_{i-1}^2 \int_0^1 \widetilde{w}(t) f''(t_i - th_{i-1})\, dt, \end{aligned}$$

where $\widetilde{w}(t) = 1 - t$. These F_i are obtained by differentiating the error functional with respect to the knots. Hence, for $f(t) = t^2$, $F_i = 0$, $i = 1, \ldots, N$, gives $t_i = i/(N+1)$, $i = 1, \ldots, N$. Also, for each i, $1 \leq i \leq N$, we obtain

$$\sum_{j=1}^N \frac{\partial F_i}{\partial t_j} = h_i^2 \int_0^1 \widetilde{w}(\tau) f'''(t_i + \tau h_i)\, d\tau - h_{i-1}^2 \int_0^1 w(\tau) f'''(t_i - \tau h_{i-1})\, d\tau.$$

Therefore, by the same argument as in the L_2 case, we have Theorems 7 and 8. To obtain Theorem 9, we just use the same function constructed in the proof of Theorem 3 and a slightly different proof.

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