MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 144
OCTOBER 1978, PAGES 1311–1316

# On an Integral Summable to $2\xi(s)/s(s-1)$

By P. L. Walker

**Abstract.** Let $\psi(x) = \Sigma_{n=1}^{\infty} e^{-n^2\pi x}$, and $\chi(u) = e^{u/2}(1 + 2\psi(e^{2u}))$. The divergent integral $2\int_0^\infty \chi(u) \cos tu\, du$ is shown to be summable for certain complex values of $t$ to the function $2\xi(s)/s(s-1) = \pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s)$, where $s = \frac{1}{2} + it$, and $\zeta(s)$ is the zeta-function of Riemann.

The values of a resulting approximation to $2\xi(s)/s(s-1)$ are computed and its zeros located.

**1. Introduction.** The formula

$$2\xi(s)/s(s-1) = \pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \psi(x)(x^{s/2} + x^{(1-s)/2})\frac{dx}{x},$$

where $\psi(x) = \Sigma_{n=1}^{\infty} e^{-n^2\pi x}$, occurs in Riemann's paper [4] in which the zeta function was first studied as a function of a complex variable.

If we restrict attention to the critical strip $\{s : 0 < \operatorname{Re} s < 1\}$, and replace the terms $-1/s$, $-1/(1-s)$ by the divergent integrals $\frac{1}{2}\int_1^\infty x^{s/2-1}\, dx$, $\frac{1}{2}\int_1^\infty x^{(1-s)/2-1} dx$ which have the respective values as Hadamard finite parts, we obtain

$$\begin{aligned} 2\xi(s)/s(s-1) &= \int_1^\infty (\tfrac{1}{2} + \psi(x))(x^{s/2} + x^{(1-s)/2})\frac{dx}{x} \\ &= 2\int_0^\infty (1 + 2\psi(e^{2u}))e^{u/2}\cos tu\, du = 2\int_0^\infty \chi(u)\cos tu\, du, \end{aligned} \tag{1.1}$$

where $x = e^{2u}$ and $s = \frac{1}{2} + it$.

The well-known identity $(1 + 2\psi(x)) = x^{-1/2}(1 + 2\psi(x^{-1}))$ shows that $\chi$ is an even function; in addition, it has natural boundaries on the lines $\operatorname{Im} u = \pm\frac{1}{4}\pi$, and increases exponentially along the real axis.

The first aim of the present paper is to show that the formula (1.1), which represents $2\xi(s)/s(s-1)$ as the (formal) Fourier cosine transform of an increasing function on $[0, \infty]$, is valid for suitable values of $t$ if we include a summability kernel in the integral.

Specifically, we prove

THEOREM 1. *Let*

$$H = \{z : \operatorname{Re} z < 0\} \cup \{z : \operatorname{Re} z \geq 0, |2(1 + \sqrt{1+z^2})^{-1}\exp(\sqrt{1+z^2} - 1)| < 1\},$$

*and* $H_1 = i(H - \frac{1}{2}) \cap \{-i(H - \frac{1}{2})\}$.

(Figure 1 shows the boundary of $H$, with $H$ lying to the left; in Figure 2, $H_1$ is the region enclosed by the curved arcs, while the dotted lines show the boundaries of the critical strip, $\operatorname{Im} t = \pm\frac{1}{2}i$.)

Received July 7, 1977.
*AMS (MOS) subject classifications* (1970). Primary 10H05, 42A68, 42–04.

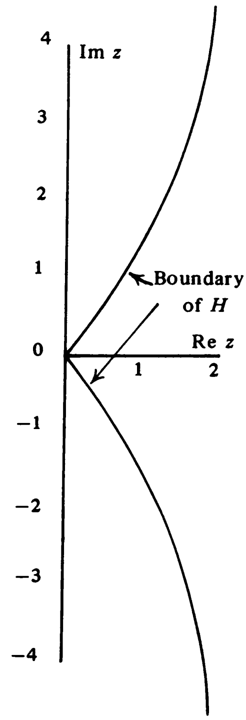


FIGURE 1. The region $H$, which lies to the left of the indicated boundary.

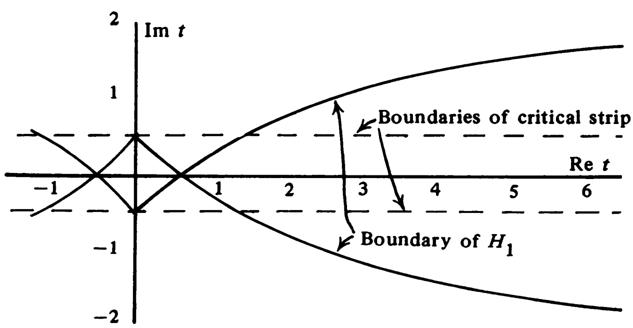


FIGURE 2. The region $H_1$, which contains the critical strip for Re $t \geqslant 2$.

*For* $n = 1, 2, 3, \ldots$, *we define* $P_n(t)$ *by*

$$P_n(t) = 2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n \chi(u) \cos tu \, du.$$

*Then* $(P_n(t))$ *is a sequence of entire functions which converges to* $2\xi(s)/s(s-1)$, $s = \frac{1}{2} + it$, *uniformly on each compact subset of* $H_1$, *as* $n \longrightarrow \infty$.

A similar result states that $G_\lambda(t) = 2\int_0^\infty e^{-\lambda u^2} \chi(u) \cos tu \, du$ converges to $2\xi(s)/s(s-1)$ as $\lambda \longrightarrow 0$ if $\pm t \in \{z : |\mathrm{Arg}(z - \frac{1}{2})| < \frac{1}{4}\pi\}$, but this result is less suited to computation.

The second aim of the paper is to investigate the zeros of $P_n(t)$. A result of Pólya and Szegö [3] proves that $P_n$ has infinitely many zeros, all but a finite number of which lie on the real axis. However, the number of nonreal zeros increases with $n$; in section 3 we calculate the zeros explicitly when $n = 10$, using a Fast Fourier Transform programme. The calculation was carried out by Dr. G. Tunnicliffe Wilson and Mr. J. Hampton whose assistance the author is pleased to acknowledge.

**2. Summability of the Integral.** In this section we prove the theorem stated in the introduction. We shall need the following lemmas, of which the first is elementary (see for instance [2, §2.4, Eq. (3)], a much more general result), and the second may be deduced from known Bessel function formulas, (see for instance [1, 9.6.18 and 9.7.7]), or directly from the method of steepest descents.

LEMMA 1. *Let $f$ be a complex valued function which is defined and continuous on* $[0, 1]$, $f(0) = 1$, $|f(t)| < 1$ *if* $0 < t \leqslant 1$. *Suppose $f$ has a finite nonzero right-hand derivative at* $0$.

*Then*

$$n\int_0^1 (f(t))^n dt \longrightarrow -[f'_+(0)]^{-1} \quad \text{as } n \longrightarrow \infty.$$

LEMMA 2. *Let* $\operatorname{Re} z > 0$. *Then as* $n \longrightarrow \infty$,

$$\int_{-n}^{n}\left(1 - \frac{t^2}{n^2}\right)e^{zt}dt = n\int_{-1}^{1}\{(1-t^2)e^{zt}\}^n dt \sim (1+z^2)^{-1/4}(2\pi n/z)^{1/2}E^n,$$

*where* $E = 2(1+\sqrt{1+z^2})^{-1}\exp(\sqrt{1+z^2}-1)$, *and the* $\sim$ *indicates that the ratio of the two sides approaches unity as* $n \longrightarrow \infty$.

*Proof of Theorem* 1. We have

$$P_n(t) = 2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n \chi(u)\cos tu\, du = 2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{u/2}(1 + 2\psi(e^{2u}))\cos tu\, du.$$

The rapid decrease of $\psi(e^{2u})$ at infinity together with the fact that $(1 - u^2/n^2)^n$ increases to unity as $n \longrightarrow \infty$, shows that

$$2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{u/2} 2\psi(e^{2u})\cos tu\, du \longrightarrow 2\int_0^\infty e^{u/2}2\psi(e^{2u})\cos tu\, du$$

for all complex $t$, as $n \longrightarrow \infty$, by dominated convergence. Hence it is sufficient to show that if $t \in H_1$, then

$$2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{u/2}\cos tu\, du = \int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{u/2}(e^{itu} + e^{-itu})du$$

approaches

$$-\frac{1}{\frac{1}{2}+it} - \frac{1}{\frac{1}{2}-it} = -\frac{1}{s} - \frac{1}{1-s}, \quad \text{as } n \longrightarrow \infty.$$

Consider first

$$\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{(\frac{1}{2}+it)u} du = \int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{su} du$$

$$= n\int_0^1 \{(1 - u^2)e^{su}\}^n du = I, \quad \text{say.}$$

If Re $s \leqslant 0, s \neq 0$, this tends to $-1/s$ as $n \longrightarrow \infty$ taking $f(u) = (1 - u^2)e^{su}$ in Lemma 1.

If Re $s > 0$, we write $I$ as

$$n\int_{-1}^1 \{(1 - u^2)e^{su}\}^n du - n\int_{-1}^0 \{(1 - u^2)e^{su}\}^n du.$$

The first term tends to zero as $n \longrightarrow \infty$ by Lemma 2, provided $|E| < 1$ where $E = 2(1 + \sqrt{1 + s^2})^{-1}\exp(\sqrt{1 + s^2} - 1)$. The second term may be written as $-n\int_0^1 \{(1 - u^2)e^{-su}\}^n du$ which tends to $-1/s$ by Lemma 1, since now Re $s > 0$. We have shown that if $s \in H$, $I \longrightarrow -1/s$ as $n \longrightarrow \infty$, and the result is easily seen to hold uniformly on compact subsets of $H$. Similarly, if $1 - s \in H$,

$$2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n e^{(1-s)u} du \longrightarrow -\frac{1}{1 - s}, \quad \text{as } n \longrightarrow \infty.$$

But the conditions $s$, $1 - s \in H$ are equivalent to $t \in H_1$, and the result is proved.

**3. Location of Zeros.** In the formula

$$P_n(t) = 2\int_0^n \left(1 - \frac{u^2}{n^2}\right)^n \chi(u)\cos tu \, du,$$

the integrand has a discontinuity in the $n$th derivative at the point $u = n$. Consequently,

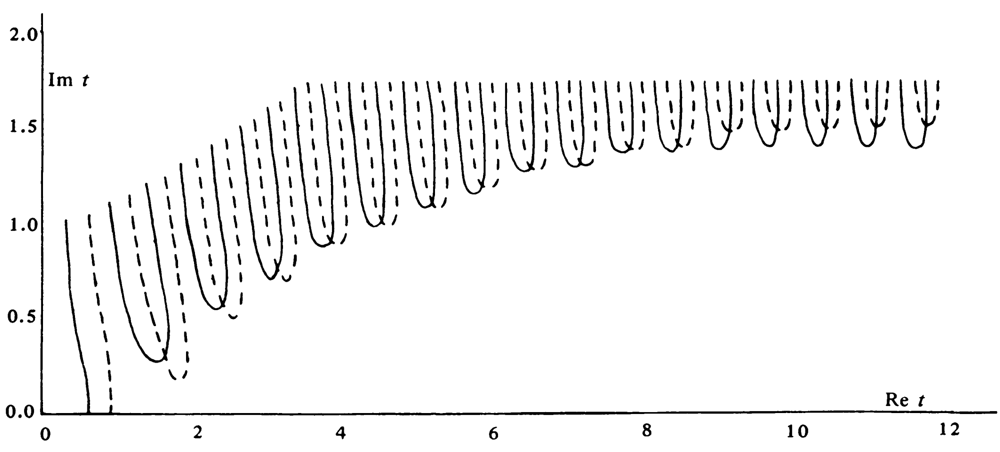


FIGURE 3(a). The zeros of $P_{10}$.
The zeros of Re $P_{10}$ are indicated by continuous lines, and those of Im $P_{10}$ are indicated by broken lines. The imaginary part is also zero on both axes.

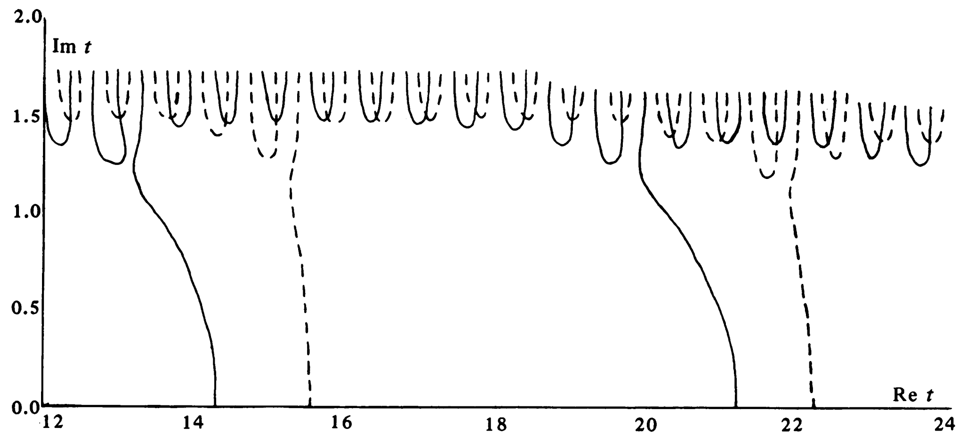


FIGURE 3(b). Continuation of 3(a) to the right.

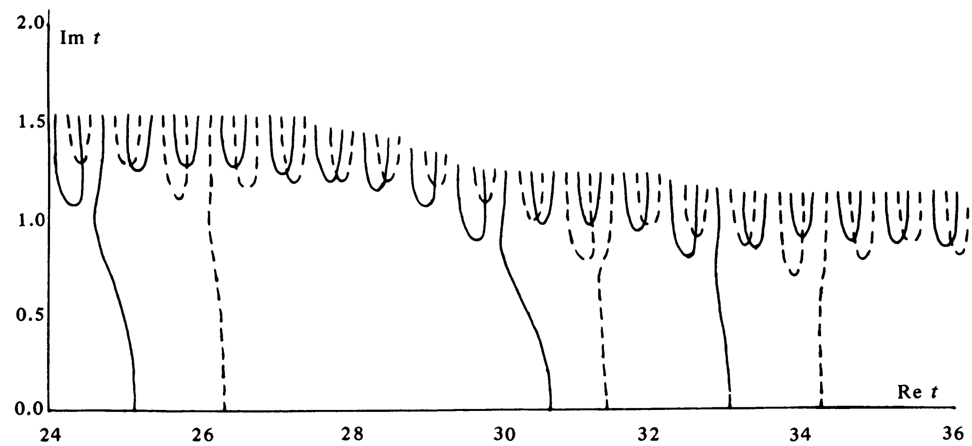


FIGURE 3(c). Continuation of 3(b) to the right.

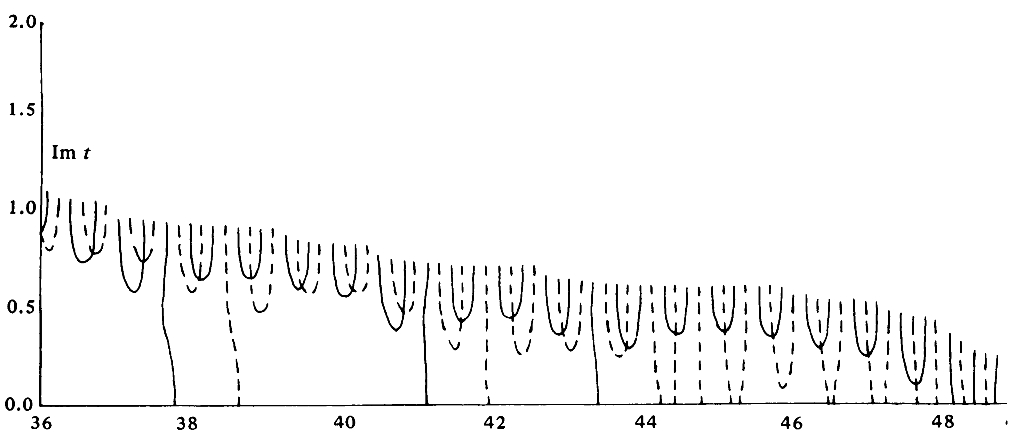


FIGURE 3(d). Continuation of 3(c) to the right.

$n$ integrations by parts show that for large $|t|$, the behavior of $P_n(t)$ is close to that of $\cos nt$ ($n$ even) or $\sin nt$ ($n$ odd). In particular, $P_n$ has infinitely many zeros, all but a finite number of which lie on the real axis—see Pólya and Szegö [3, Vol. I, solution to Q199, §III].

Inside the region $H_1$ of Theorem 1, $P_n$ converges uniformly to $\pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s)$ so that the nontrivial zeros of the zeta-function are left behind in the critical strip while the other zeros (of which, for large $K$, there are approximately $2K\pi/n$ inside a circle of radius $K$) lie in a roughly arch-shaped configuration above and below the real axis.

The computation of $P_{10}(t)$ was carried out over a range of $t$, $0 \leqslant \operatorname{Re} t\ (0.1) \leqslant 50$, and $0 \leqslant \operatorname{Im} t\ (0.1) \leqslant 2$, using a radix 2 Fast Fourier Transform programme of order 2048, applied to the integrand sampled over the range $[-10, 10]$. Double precision arithmetic was used since for $t$ near 50, $P_{10}(t)$ is of the order of $10^{-17}$. The position of the zeros of the real part of $P_{10}(t)$ is shown by the continuous curves in Figure 3, and the zeros of the imaginary part by the broken curves. The figure shows 12 zeros on the real axis with $0 < \operatorname{Re} t < 49$, and 69 conjugate complex pairs.

Department of Mathematics
University of Lancaster
Bailrigg, Lancaster LA1 4YL, England

1. M. ABRAMOWITZ & I. A. STEGUN, *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*, Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1965.

2. A. ERDÉLYI, *Asymptotic Expansions*, Dover, New York, 1956.

3. G. PÓLYA AND G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1925.

4. B. RIEMANN, "Über die Anzahl der Primzahlen unter einer gegebenen Grosse," *Collected Works*, Dover, New York, 1953.