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A Proof of Convergence and an Error Bound for the Method of Bisection in $\mathbf{R}^n$

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Abstract. Let $S = \langle X_0, \ldots, X_m \rangle$ be an m-simplex in $\mathbf{R}^n$. We define "bisection" of S as follows. We find the longest edge $\langle X_i, X_j \rangle$ of S, calculate its midpoint $M = (X_i + X_j)/2$, and define two new m-simplexes S_1 and S_2 by replacing X_i by M or X_j by M.

Suppose we bisect S_1 and S_2, and continue the process for p iterations. It is shown that the diameters of the resulting simplexes are no greater then $(\sqrt{3}/2)^{\lfloor p/m \rfloor}$ times the diameter of the original simplex, where $\lfloor p/m \rfloor$ is the largest integer less than or equal to p/m.

1. Introduction and Summary. Recently devised methods for computing roots of a continuous map defined on a simplex (generalized triangle) in $\mathbf{R}^n$ involve a technique of subdivision termed "generalized bisection" ([7], [3], [4]), in which two new simplexes of comparable diameters are formed from the original simplex. An as yet unanswered question concerning such generalized bisections has been: How fast do the diameters of the resulting simplexes tend to zero, as repeated bisection is performed?

In this paper we first define bisection of an m-simplex in $\mathbf{R}^n$ and clarify the problem of convergence of the resulting method of bisection. We then prove that, after p repeated bisections, the diameters of the resulting m-simplexes in $\mathbf{R}^n$ are no greater than $(\sqrt{3}/2)^{\lfloor p/m \rfloor}$ times the diameter of the original simplex, where $\lfloor p/m \rfloor$ is the largest integer less than p/m.

2. Notation, Definitions, and Preliminary Concepts.

2.1 *Definition.* Suppose $X_0, \ldots, X_m$ are any $m + 1$ points in $\mathbf{R}^n$ $(1 \leqslant m \leqslant n)$ and suppose that $\{X_i - X_0\}_{i=1}^m$ is a linearly independent set of vectors in $\mathbf{R}^n$. Then the closed convex hull of $X_0, \ldots, X_m$, denoted $S = \langle X_0, \ldots, X_m \rangle$ is called an *m-simplex in* $\mathbf{R}^n$, while the points $X_0, \ldots, X_m$ are called the *vertices of* S ([1], [2], [3], [4], etc.).

For example, a 3-simplex in $\mathbf{R}^3$ is a tetrahedron, a 2-simplex in $\mathbf{R}^2$ is a triangle, while a 1-simplex in $\mathbf{R}^n$ is a line segment in $\mathbf{R}^n$.

2.2 *Remark.* The order in which the points $\{X_i\}_{i=0}^m$ are written in the lists $\langle X_0, \ldots, X_m \rangle$ for S determines an *orientation* of S ([1], [2], [3], etc.). However, the actual closed convex hull is independent of that order; for this reason, for results in this paper we may permute the vertices of S.

2.3 *Definition.* If $S = \langle X_0, \ldots, X_m \rangle$ is an m-simplex in $\mathbf{R}^n$, then we will call each 1-simplex $\langle X_i, X_j \rangle$, $0 \leqslant i < j \leqslant n$, an *edge* of S.

2.4 *Definition.* If S is an m-simplex in $\mathbf{R}^n$, then the *diameter* of S is equal to the quantity $\max_{X, Y \in S} \|X - Y\|_2$.

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2.5 *Remark.* The diameter of the 1-simplex $\langle A, B\rangle$ is equal to the length $\|B - A\|_2$. Furthermore, by convexity, the diameter of an arbitrary m-simplex S is equal to the maximum of the lengths of its edges.

Given an m-simplex $S = \langle X_0, \ldots, X_m\rangle$, perhaps the lengths of more than one of the $\binom{m+1}{2}$ edges $\{\langle X_i, X_j\rangle\}_{0\leqslant i<j\leqslant m}$ of S are equal to the diameter of S. However, there is a unique such edge $\langle X_k, X_{k'}\rangle$ if we require that, if the length of $\langle X_i, X_j\rangle$ is also equal to the diameter of S, then $k \leqslant i$ and $k' \leqslant j$.

2.6 *Definition.* The edge $\langle X_k, X_{k'}\rangle$ described in the preceding paragraph will be called the *selected edge* of S.

We now present the definition of bisection.

2.7 *Definition* ([7], [3]). Suppose $S_0 = \langle X_0, \ldots, X_m\rangle$ is an m-simplex, $\langle X_k, X_{k'}\rangle$ is the selected edge of S_0, and $A = (X_k + X_{k'})/2$ is the midpoint of $\langle X_k, X_{k'}\rangle$. Then two new simplexes

$$S_1 = \langle X_0, \ldots, X_{k-1}, A, X_{k+1}, \ldots, X_{k'}, \ldots, X_m\rangle$$

and

$$S_2 = \langle X_0, \ldots, X_k, \ldots, X_{k'-1}, A, X_{k'+1}, \ldots, X_m\rangle$$

may be formed such that the interiors of S_1 and S_2 are disjoint and $S_0 = S_1 \cup S_2$. We call S_1 the *lower simplex* from S_0, and we call S_2 the *upper simplex* from S_0. The process of producing S_1 and S_2 is called *bisection* of S_0, while the ordered pair (S_1, S_2) is called *the bisection* of S_0.

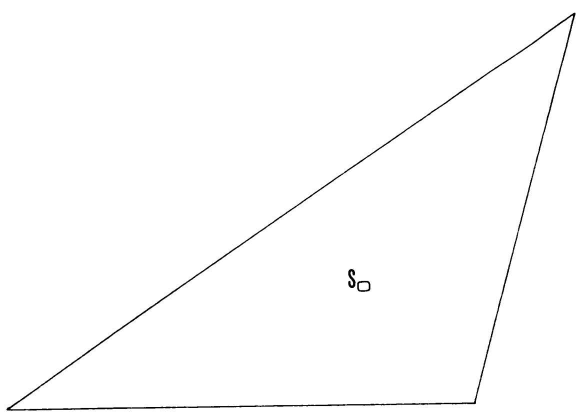

FIGURE 2.1

Given an m-simplex S, we will bisect S, bisect the elements of the bisection of S, and continue the process to get a sequence of sets of simplexes related through bisection ([7], [3]). Figures 2.1–2.3 illustrate one such sequence of sets of 2-simplexes in $\mathbf{R}^2$. There, S_0 is the original simplex, (S_1, S_2) is the bisection of S_0, and (S_3, S_4) is the bisection of S_2; S_3 and S_4 are each two bisections removed from S_0, while S_1 and S_2 are only one bisection removed from S_0. We say that S_3 is *produced after two bisections* of S_0, S_1 is *produced after one bisection* of S_0, etc.

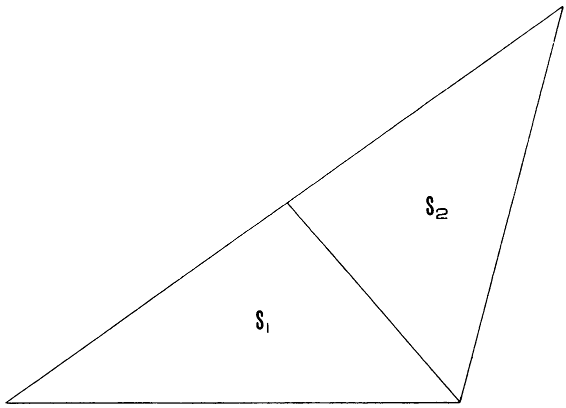

FIGURE 2.2

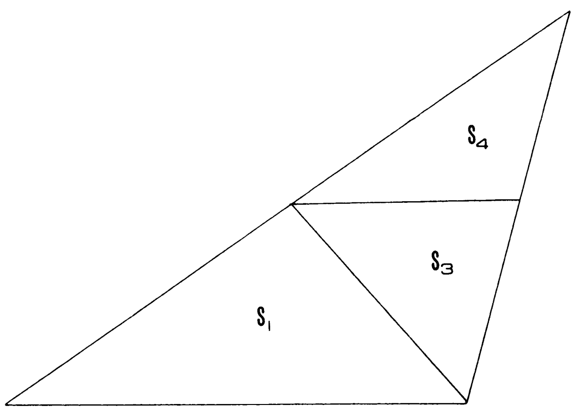

FIGURE 2.3

Having defined the above concepts, we can present the main theorem.

3. The Convergence Theorem and Proof.

3.1 THEOREM. *Let S_0 be an m-simplex, let p be any positive integer, and let S_p be any m-simplex produced after p bisections of S_0. Then the diameter of S_p is no more than $(\sqrt{3}/2)^{\lfloor p/m \rfloor}$ times the diameter of S_0, where $\lfloor p/m \rfloor$ is the largest integer less than or equal to p/m.*

3.2 COROLLARY. *If S_0 is an n-simplex in $\mathbf{R}^n$, and S_p is any simplex produced after p bisections of S_0, then the diameter of S_p is no greater than $(\sqrt{3}/2)^{\lfloor p/n \rfloor}$ times the diameter of S_0.*

3.3 *Proof of Theorem* 3.1. It suffices to show that, if $p = m$, then the diameter of S_p is no greater than $\sqrt{3}/2$ times the diameter of S_0, so assume $p = m$. Then there is a sequence of simplexes S_q, $q = 1, \ldots, m$, such that S_1 is produced from bisection of S_0, S_q is produced from bisection of S_{q-1} for $1 < q < m$, and S_m is produced from bisection of S_{m-1}. With the sequence $\{S_q\}_{q=0}^m$ so defined, we set D_q equal to the diameter of $S_q = \langle X_0^{(q)}, \ldots, X_m^{(q)} \rangle$, we let $\langle X_{k_q}^{(q)}, X_{k'_q}^{(q)} \rangle$ be the selected edge of S_q, and we set $d_{i,j}^{(q)}$ equal to the length of the edge $\langle X_i^{(q)}, X_j^{(q)}\rangle$, for $0 \leqslant i < j \leqslant m$.

We may switch the labels of k_q and k'_q if necessary so that $X_{k_q}^{(q+1)} = (X_{k_q}^{(q)} + X_{k'_q}^{(q)})/2$ and $X_i^{(q+1)} = X_i^{(q)}$ for $0 \leqslant i \leqslant m$ and $i \neq k_q$, for $q = 0, \ldots, m-1$.

We will prove Theorem 3.1 in the setting outlined above by showing that at least $m + (m-1) + \cdots + (m-q+1)$ of the $m(m+1)/2$ distinct edges of S_q each have length no greater than $(\sqrt{3}/2)D_0$, for $1 \leqslant q \leqslant m$. The proof will proceed by induction on q. Lemma 3.4 (infra) is central to the argument.

If $q = 1$, we invoke (i), Lemma 3.4 to get that each of the m lengths

$$d_{0,k_0}^{(1)}, \ldots, d_{k_0-1,k_0}^{(1)}, d_{k_0,k_0+1}^{(1)}, \ldots, d_{k_0,m}^{(1)} \tag{3.1}$$

is less than $(\sqrt{3}/2)D_0$. Furthermore, unless $D_1 \leqslant (\sqrt{3}/2)D_0$ (in which case the conclusion of Theorem 3.1 follows by (iv), Lemma 3.4), we have $k_1 \neq k_0$, and the $m-1$ edges

$$[\langle X_i^{(2)}, X_{k_1}^{(2)}\rangle_{0\leqslant i<k_1, i\neq k_0}] \cup [\langle X_{k_1}^{(2)}, X_i^{(2)}\rangle_{k_1<i\leqslant m, i\neq k_0}] \tag{3.2}$$

are distinct from the m edges

$$[\langle X_i^{(2)}, X_{k_0}^{(2)}\rangle = \langle X_i^{(1)}, X_{k_0}^{(1)}\rangle_{0\leqslant i<k_0}] \cup [\langle X_{k_0}^{(2)}, X_i^{(2)}\rangle = \langle X_{k_0}^{(1)}, X_i^{(1)}\rangle_{k_0<i\leqslant m}], \tag{3.3}$$

where the equalities in (3.3) follow from the definition of bisection. Moreover, application of (i) and (iv), Lemma 3.4 shows that each edge in (3.2) also has a length of at most $(\sqrt{3}/2)D_1 \leqslant (\sqrt{3}/2)D_0$.

To complete the induction we assume that after q bisections ($q < m$), the $m + (m-1) + \cdots + (m-q+1)$ distinct edges in the set

$$\left\{\bigcup_{j=0}^{q-1} [\langle X_i^{(q)}, X_{k_j}^{(q)}\rangle_{0\leqslant i<k_j, i\neq k_0,\ldots,k_{j-1}}]\right\} \cup \left\{\bigcup_{j=0}^{q-1} [\langle X_{k_j}^{(q)}, X_i^{(q)}\rangle_{k_j<i\leqslant m, i\neq k_0,\ldots,k_{j-1}}]\right\}, \tag{3.4}$$

where $k_i \neq k_j$ for $0 \leqslant i < j \leqslant q-1$, each have length no greater than $(\sqrt{3}/2)D_0$. Then, unless $D_q \leqslant (\sqrt{3}/2)D_0$ (which would imply the conclusion of Theorem 3.1 by repeated application of (iv), Lemma 3.4), we have

$$k_q \notin \{k_0, \ldots, k_{q-1}\}. \tag{3.5}$$

Therefore, the $m - q$ edges

$$[\langle X_i^{(q+1)}, X_{k_q}^{(q+1)}\rangle_{0\leqslant i<k_q, i\neq k_0,\ldots,k_{q-1}}] \cup [\langle X_{k_q}^{(q+1)}, X_i^{(q+1)}\rangle_{k_q<i\leqslant m, i\neq k_0,\ldots,k_{q-1}}] \tag{3.6}$$

are distinct from the $m + (m-1) + \cdots + (m-q+1)$ edges in the set

$$\left\{ \bigcup_{j=0}^{q-1} [\langle X_i^{(q+1)}, X_{k_j}^{(q+1)}\rangle = \langle X_i^{(q)}, X_{k_j}^{(q)}\rangle_{0 \le i < k_j, i \ne k_0, \ldots, k_{j-1}}] \right\} \cup \left\{ \bigcup_{j=0}^{q-1} [\langle X_{k_j}^{(q+1)}, X_i^{(q+1)}\rangle = \langle X_{k_j}^{(q)}, X_i^{(q)}\rangle_{k_j < i \le m, i \ne k_0, \ldots, k_{j-1}}] \right\}, \tag{3.7}$$

where the equalities in (3.7) follow from the definition of bisection. Moreover, we apply (i) of Lemma 3.4, with S_{q+1} replacing S_1 and S_q replacing S_0, then apply (iv), Lemma 3.4 repeatedly to show that each of the edges in (3.6) has length less than

$$\frac{\sqrt{3}}{2} D_{q-1} \le \frac{\sqrt{3}}{2} D_{q-2} \le \cdots \le \frac{\sqrt{3}}{2} D_1 \le \frac{\sqrt{3}}{2} D_0.$$

Furthermore, we apply the induction hypothesis to each of the edges in (3.7) to show that each of those also has length of at most $(\sqrt{3}/2)D_0$.

Therefore, by induction, after m bisections, there are $m + (m-1) + \cdots + 1 = m(m+1)/2$ distinct edges in S_m whose lengths are at most $(\sqrt{3}/2)D_0$. But an m-simplex has precisely $m(m+1)/2$ distinct edges, so by Remark 2.5, $D_m \le (\sqrt{3}/2)D_0$.

Repeating the above argument $\lfloor p/m \rfloor$ times with S_m and D_m replacing S_0 and D_0, respectively, gives $D_p \le (\sqrt{3}/2)^{\lfloor p/m \rfloor} D_0$ for arbitrary integers p; this is the conclusion of Theorem 3.1.

3.4 LEMMA. *Let $S_0 = \langle X_0, \ldots, X_m \rangle$ be any m-simplex in $\mathbf{R}^n$, suppose $\langle X_k, X_{k'} \rangle$ ($\langle X_{k'}, X_k \rangle$) is the selected edge of S_0, and suppose*

$$S_1 = \left\langle X_0, \ldots, X_{k-1}, \frac{X_k + X_{k'}}{2}, X_{k+1}, \ldots, X_m \right\rangle$$

is the lower (the upper) simplex from S_0. Suppose further that $d_{i,j}^{(0)}$ and $d_{i,j}^{(1)}$ are defined as in Theorem 3.1, for $0 \le i < j \le m$; suppose that $D_0 = d_{k,k'}^{(0)}$, ($D_0 = d_{k',k}^{(0)}$) is the diameter of S_0 and suppose that D_1 is the diameter of S_1. Then

(i) *$d_{i,k}^{(1)} \le (\sqrt{3}/2)D_0$ and $d_{k,i}^{(1)} \le (\sqrt{3}/2)D_0$ for $0 \le i < k$ and for $k < i \le m$, respectively*;

(ii) *$d_{k,k'}^{(1)} = d_{k,k'}^{(0)}/2$*;

(iii) *$d_{i,j}^{(1)} = d_{i,j}^{(0)}$ for $0 \le i < j \le m$ and $i \ne k$*;

(iv) *$D_1 \le D_0$.*

3.5 *Proof of Lemma* 3.4. We observe that both (ii) and (iii) follow directly from the definition of bisection. Furthermore, (iv) follows from (i), (ii), and (iii), so we need only prove (i).

To prove (i), we set $X_j = (x_{j,1}, \ldots, x_{j,n})$ where $x_{j,p} \in \mathbf{R}$ for $1 \le p \le n$ and $0 \le j \le m$. Then

$$(d_{j,k}^{(1)})^2 = \sum_{p=1}^{n} \left(x_{j,p} - \frac{x_{k,p} + x_{k',p}}{2} \right)^2 \quad \text{for } 0 \le j < k \tag{3.8}$$

and

$$(d_{k,j}^{(1)})^2 = \sum_{p=1}^{n} \left(x_{j,p} - \frac{x_{k,p} + x_{k',p}}{2} \right)^2 \quad \text{for } k < j \le m. \tag{3.9}$$

But, for $1 \leqslant p \leqslant n$ we have

$$\left(x_{j,p} - \frac{x_{k,p} + x_{k',p}}{2}\right)^2 = x_{j,p}^2 - x_{j,p}x_{k,p} - x_{j,p}x_{k',p} + \frac{x_{k,p}^2}{4} + \frac{x_{k,p}x_{k',p}}{2} + \frac{x_{k',p}^2}{4}$$

$$(3.10) \qquad = \left(\frac{x_{j,p}^2}{2} - x_{j,p}x_{k,p} + \frac{x_{k,p}^2}{2}\right) + \left(\frac{x_{j,p}^2}{2} - x_{j,p}x_{k',p} + \frac{x_{k',p}^2}{2}\right) - \left(\frac{x_{k,p}^2}{4} - \frac{x_{k,p}x_{k',p}}{2} + \frac{x_{k',p}^2}{4}\right)$$

$$= \frac{(x_{j,p} - x_{k,p})^2}{2} - \frac{(x_{k,p} - x_{k',p})^2}{4} + \frac{(x_{j,p} - x_{k',p})^2}{2}.$$

Hence, summing (3.10) over all p gives

$$(3.11) \qquad (d_{k,j}^{(1)})^2 = \left[\sum_{p=1}^{n} (x_{j,p} - x_{k,p})^2\right]\Big/2 + \left[\sum_{p=1}^{n} (x_{j,p} - x_{k',p})^2\right]\Big/2 - \left[\sum_{p=1}^{n} (x_{k,p} - x_{k',p})^2\right]\Big/4$$

$$= [(d_{k,j}^{(0)})^2 + (d_{j,k'}^{(0)})^2]/2 - (d_{k,k'}^{(0)})^2/4,$$

for $k < j \leqslant m$ (we have the same inequality with $d_{j,k}^{(1)}$ replacing $d_{k,j}^{(1)}$ for $0 \leqslant j < k$). However, by assumption we have

$$(3.12) \qquad (d_{k,k'}^{(0)})^2 = D_0^2$$

and

$$(3.13) \qquad \begin{aligned} &(d_{j,k}^{(0)})^2 \leqslant D_0^2, \quad 0 \leqslant j < k; \qquad (d_{k,j}^{(0)})^2 \leqslant D_0^2, \quad k < j \leqslant m; \\ &(d_{j,k'}^{(0)})^2 \leqslant D_0^2, \quad 0 \leqslant j < k'; \qquad (d_{k',j}^{(0)})^2 \leqslant D_0^2, \quad k' < j \leqslant m. \end{aligned}$$

Combining (3.11), (3.12), and (3.13) gives

$$(3.14) \qquad (d_{k,j}^{(1)})^2 \leqslant \frac{(D_0^2 + D_0^2)}{2} - \frac{D_0^2}{4} = \frac{3}{4}D_0^2.$$

Taking square roots of both sides gives

$$(3.15) \qquad d_{k,j}^{(1)} \leqslant \frac{\sqrt{3}}{2} D_0, \qquad k < j \leqslant m.$$

To complete the proof of (i), Lemma 3.4, we observe that (3.15) holds when we replace the left member by $d_{j,k}^{(1)}$ for $0 \leqslant j < k$.

It can be seen that the bound given in Theorem 3.1 is sharp for $p = m$. To show this we set $m = n - 1$ and let S be the $(n - 1)$-simplex whose ith vertex is the ith coordinate vector in $\mathbf{R}^n$. However, numerical experiments verify that, for the same simplex S, the diameters are reduced by a factor of 2 every m iterations for $p > m$.

An important related problem is to determine a nonzero lower bound on the ratio of the lengths of the smallest edges of simplexes to the lengths of the largest edges, as bisections are performed. Since the area of each m-simplex is reduced by a factor of 2 by a simple bisection, such a bound may give a better estimate of the rate of convergence. This problem has been solved for triangles ($n = m = 2$) [5].

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