Energy Conserving Norms for the Solution of Hyperbolic Systems of Partial Differential Equations

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Abstract. The problem of finding an energy conserving norm for the solution of the hyperbolic system of partial differential equations $\partial u/\partial t = A \, \partial u/\partial x$, subject to boundary conditions, is reduced to the problem of characterizing those matrices appearing in the boundary conditions which satisfy two specific matrix equations. Necessary and sufficient conditions on the coefficient matrix A and the matrices appearing in boundary conditions are derived for an energy conserving norm to exist. Thus, these conditions serve as tests on a given system which determine whether or not the solution will have its energy conserved in some norm. In addition, some examples of specific systems and boundary conditions are provided.

I. Motivation. Consider the hyperbolic system of partial differential equations

$$(1.1) \qquad \frac{\partial u}{\partial t} = A \, \frac{\partial u}{\partial x} \quad \text{for } t > 0 \text{ and } 0 < x < 1,$$

where A is a real constant square $n \times n$ matrix. The hyperbolicity of the system (1.1) implies there exists a nonsingular real (constant) matrix T which diagonalizes A, i.e.

$$(1.2) \qquad TAT^{-1} = \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\},$$

where the λ_j's, which are the eigenvalues of A, are real. Then the system (1.1) can be transformed into the diagonal system

$$(1.3) \qquad \frac{\partial w}{\partial t} = \Lambda \, \frac{\partial w}{\partial x} \quad \text{for } t > 0 \text{ and } 0 < x < 1,$$

where $w = Tu$. Without any loss of generality we may order the eigenvalues λ_j in a nonincreasing sequence, i.e. $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Suppose that p of these eigenvalues are positive and q are negative and that $p + q = n$ (we, therefore, exclude the possibility of vanishing eigenvalues); then, we may perform the partitions

$$(1.4) \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where Λ_1 and Λ_2 are positive $p \times p$ and $q \times q$ diagonal matrices, respectively, and w_1 and w_2 are p and q vectors, respectively. Well-posed linear homogeneous boundary conditions for the systems (1.1) or (1.3) can always be written in the form (see Kreiss and Oliger [5])

$$(1.5) \qquad w_2(t, 0) = S_2 w_1(t, 0) \quad \text{and} \quad w_1(t, 1) = S_1 w_2(t, 1),$$

where S_1 and S_2 are $p \times q$ and $q \times p$ matrices, respectively.

Given a symmetric positive definite matrix E, the E-energy is defined to be

$$(1.6) \qquad E(t) = \frac{1}{2} \int_0^1 (u^t E u)\, dx.$$

The availability of *a priori* estimates for E is important in the analysis of both analytical and numerical solutions of the system (1.1), e.g. see Gunzburger [3]. If u is a solution of the system (1.1), then

$$(1.7) \qquad \frac{dE}{dt} = \frac{1}{2} \int_0^1 (u_x^t A^t E u + u^t E A u_x)\, dx.$$

In order to make the integrand in (1.7) an exact differential, we require that E symmetrizes A, i.e.

$$(1.8) \qquad EA = A^t E$$

so that

$$(1.9) \qquad \frac{dE}{dt} = \frac{1}{2} [u^t(t, 1)EAu(t, 1) - u^t(t, 0)EAu(t, 0)].$$

The existence of a positive definite symmetrizer E for a matrix A has been established by Taussky [6] for exactly those matrices A for which the system (1.1) is hyperbolic i.e., those matrices A similar to a real symmetric matrix. Then combining (1.2) and (1.8) yields

$$(1.10) \qquad (T^t)^{-1} E T^{-1} \Lambda = \Lambda (T^t)^{-1} E T^{-1},$$

so that $(T^t)^{-1} E T^{-1}$ commutes with a diagonal matrix. If the eigenvalues λ_j of A are distinct (the case of repeated eigenvalues is considered later), this implies that

$$(1.11) \qquad E = T^t D T,$$

where D is a positive diagonal matrix. Equation (1.11) relates the symmetrizing matrix E to the diagonalizing matrix T. Substitution of (1.11) into (1.6) yields

$$E(t) = \frac{1}{2} \int_0^1 (w^t D w)\, dx;$$

and if we partition D as Λ was in (1.4), i.e.

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

(1.2), (1.4), (1.5), (1.9) and (1.11) yield

$$(1.12) \quad dE/dt = \tfrac{1}{2}\{w_2^t(t, 1)[S_1^t D_1 \Lambda_1 S_1 - D_2 \Lambda_2] w_2(t, 1)$$
$$+ w_1^t(t, 0)[S_2^t D_2 \Lambda_2 S_2 - D_1 \Lambda_1] w_1(t, 0)]\}.$$

The purpose of this work is to determine what conditions the matrices A, S_1 and S_2 must satisfy in order that a matrix E, or equivalently D, exists such that the E-energy is conserved, i.e.

$$(1.13) \qquad\qquad dE/dt = 0,$$

for arbitrary initial conditions, or equivalently, for arbitrary values of $w_1(t, 0)$ and $w_2(t, 1)$. Then comparison of (1.12) and (1.13) yields that in order for the E-energy to be conserved, we must have

$$(1.14) \qquad S_1^t D_1 \Lambda_1 S_1 - D_2 \Lambda_2 = 0 \quad \text{and} \quad S_2^t D_2 \Lambda_2 S_2 - D_1 \Lambda_1 = 0.$$

Clearly, these conditions are also sufficient.

We collect the main results of this section in the following lemmas:

LEMMA 1. *Let A have distinct real eigenvalues and have the decomposition* (1.2). *Then E symmetrizes A if and only if* (1.11) *holds for some positive diagonal matrix D.*

LEMMA 2. *If E symmetrizes A and if the E-energy is defined by* (1.6), *then the E-energy of the system* (1.1) *subject to the boundary conditions* (1.5) *is conserved if and only if* (1.14) *holds.*

II. Characterizations of Coefficient and Boundary Matrices. Since D_1, D_2, Λ_1 and Λ_2 are all positive diagonal matrices, we may define the positive diagonal matrices B_1^2 and B_2^2 by

$$(2.1) \qquad\qquad B_1^2 = D_1 \Lambda_1 \quad \text{and} \quad B_2^2 = D_2 \Lambda_2.$$

Substituting into (1.14) yields

$$(2.2) \qquad\qquad S_1^t B_1^2 S_1 = B_2^2 \quad \text{and} \quad S_2^t B_2^2 S_2 = B_1^2$$

as necessary and sufficient conditions for the conservation of E-energy.

In order to characterize A, S_1 and S_2 we shall make use of the following lemma:

LEMMA 3. *Let U and V be $p \times q$ and $q \times p$ matrices, respectively, and suppose $U^t FU = G$ and $V^t GV = F$, where F and G are nonsingular. Then, $p = q$ and U and V are nonsingular.*

Proof. First, note that G is $q \times q$ and F is $p \times p$. Since the rank of the product of matrices is less than or equal to the rank of each factor, from $U^t FU = G$ it follows that rank $U = q$ and $q \leqslant p$; and from $V^t GV = F$ it follows that rank $V = p$ and $p \leqslant q$. Therefore $p = q$ and U and V are nonsingular. $\square$

We begin the characterization of A, S_1 and S_2 by:

THEOREM 1. *If S_1 and S_2 satisfy* (1.14), *or equivalently* (2.2), *then they are square of the same size and nonsingular. As a result, if A has distinct real eigenvalues, then necessary conditions for the conservation of the E-energy (where E symmetrizes A) of the solution $u(x, t)$ to* (1.1) *and* (1.5) *are:*

(1) *A must have an even order n and the same number of positive and negative eigenvalues; and*

(2) *The boundary condition matrices in* (1.5) *must be square of order n/2 and nonsingular.*

Proof. This theorem is an immediate consequence of Lemmas 1, 2, and 3, and the nonsingularity of B_1^2 and B_2^2.   □

Now that the necessity of S_1 and S_2 being nonsingular is established, definite conditions on S_1 and S_2 such that the E-energy is conserved are given in:

THEOREM 2. *Let A have distinct real eigenvalues. Then the E-energy (where E symmetrizes A) of the solution u(x, t) to* (1.1) *and* (1.5) *is conserved if and only if S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices, that is, there exist positive diagonal matrices C_1 and C_2 such that $C_1 S_1 C_2$ and $C_1 S_2^{-1} C_2$ are orthogonal matrices.*

Proof. Necessary and sufficient conditions for the conservation of E-energy are that (2.2) holds. Clearly, (2.2) may be written as

$$(B_1 S_1 B_2^{-1})^t (B_1 S_1 B_2^{-1}) = I \quad \text{and} \quad (B_1 S_2^{-1} B_2^{-1})^t (B_1 S_2^{-1} B_2^{-1}) = I.$$

Then the choice

$$C_1 = B_1 = (D_1 \Lambda_1)^{\frac{1}{2}} \quad \text{and} \quad C_2 = B_2^{-1} = (D_2 \Lambda_2)^{-\frac{1}{2}}$$

yields that $C_1 S_1 C_2$ and $C_1 S_2^{-1} C_2$ are orthogonal matrices.

On the other hand, suppose that positive diagonal matrices C_1 and C_2 exist such that $C_1 S_1 C_2$ and $C_1 S_2^{-1} C_2$ are orthogonal matrices. Then the choice

(2.3) $D_1 = C_1^2 \Lambda_1^{-1} \quad \text{and} \quad D_2 = C_2^{-2} \Lambda_2^{-1}$

will yield (2.2) and, therefore, the E-energy will be conserved.   □

We summarize the above results in the following characterizations:

COROLLARY 2.1. *Let the coefficient matrix A of the hyperbolic system* (1.1) *have nonvanishing distinct real eigenvalues. Further, let A have an equal number of positive and negative eigenvalues. Then, the E-energy of the solution u(x, t) of* (1.1) *and* (1.5) *is conserved if and only if the matrices S_1 and S_2 appearing in* (1.5) *belong to the class of nonsingular matrices for which S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices.*

COROLLARY 2.2. *Let the matrices S_1 and S_2 which appear in the boundary conditions* (1.5) *be square, nonsingular, and have the property that S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices. Then the E-energy of the solution u(x, t) of* (1.1) *and* (1.5) *is conserved for any coefficient matrix A with nonvanishing real eigenvalues which has an equal number of positive and negative eigenvalues.*

Proofs. The first corollary is a restatement of Theorems 1 and 2. The second corollary is a consequence of (2.3) from which, given the scalings C_1 and C_2, the positive diagonal matrices Λ_1 and Λ_2 may be chosen arbitrarily due to the arbitrariness of D_1 and D_2. Then, choosing arbitrary nonsingular matrices T, we may, by the use of

(1.2) and (1.4), construct any matrix A having the properties noted in the corollary. $\square$

Remark. The matrix A constructed by the process described in the proof of Corollary (2.2) need not have distinct eigenvalues since the positive diagonal matrices Λ_1 and Λ_2 can be arbitrarily chosen. $\square$

A final consequence of the results obtained above is the following construction:

COROLLARY 2.3. *Given the matrices A, S_1 and S_2 with the necessary properties for the conservation of E-energy (these properties are given in the hypotheses of the preceding corollaries) and given the positive diagonal matrices C_1 and C_2 which simultaneously scale S_1 and S_2^{-1} to orthogonal matrices; then a matrix E for which the E-energy is conserved is given by*

$$(2.4) \qquad E = T^t \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} T,$$

where T is a diagonalizing matrix for A which groups the positive and negative eigenvalues of A together in the matrix Λ given in (1.2) and where D_1 and D_2 are given by (2.3).

Proof. The result (2.4) is an immediate consequence of Lemma 1 and Theorem 2. $\square$

Remark. For given A, S_1 and S_2, the matrix E will not be unique since the positive diagonal scalings cannot be unique and furthermore the diagonalizing matrix T is not unique. $\square$

In view of Theorem 2 and its corollaries, there is interest in investigating pairs of matrices S_1 and S_2 that can be scaled as in Theorem 2. Conditions on S_1 and S_2 such that this is possible are given in the following theorem:

THEOREM 3. *Let S_1 and S_2 be square nonsingular matrices. Then $C_1S_1C_2$ and $C_1S_2^{-1}C_2$ are orthogonal matrices for positive diagonal matrices C_1 and C_2 if and only if*

$$(2.5) \qquad S_1^{-1} = C_2^2 S_1^t C_1^2 \quad and \quad S_2^{-1} = C_1^{-2} S_2^t C_2^{-2}.$$

Moreover, in this case $C_1S_1S_2C_1^{-1}$ and $C_2^{-1}S_2S_1C_2$ are orthogonal matrices so that the eigenvalues of S_1S_2 and S_2S_1 are on the unit circle and

$$\det(S_1) = \pm 1/\det(S_2).$$

Proof. Let $Q_1 = C_1S_1C_2$ and $Q_2 = C_1S_2^{-1}C_2$ be orthogonal matrices. It follows that

$$S_1^{-1} = C_2 Q_1^t C_1 = C_2(C_2 S_1^t C_1)C_1 = C_2^2 S_1^t C_1^2$$

and

$$S_2^{-1} = C_1 Q_2 C_2^{-1} = (C_2 Q_2^t C_1)^{-1} = C_1^{-2} S_2^t C_2^{-2}.$$

On the other hand, suppose that (2.5) holds. Then, $(C_1S_1C_2)^{-1} = (C_1S_1C_2)^t$ and

$(C_1 S_2^{-1} C_2)^{-1} = (C_1 S_2^{-1} C_2)^t$ so that $C_1 S_1 C_2$ and $C_1 S_2^{-1} C_2$ are orthogonal matrices. The remainder of the proof then follows from the identities

$$C_1 S_1 S_2 C_1^{-1} = C_1 S_1 C_2 C_2^{-1} S_2 C_1^{-1} = Q_1 Q_2^t$$

and

$$C_2^{-1} S_2 S_1 C_2 = C_2^{-1} S_2 C_1^{-1} C_1 S_1 C_2 = Q_2^t Q_1. \quad \square$$

Matrices S_1 and S_2 satisfying (2.5) with C_1 an arbitrary nonsingular matrix and $C_2 = C_1^t$ have been studied and characterized by DePrima and Johnson [2] and Fan [3]. However, we have found no studies in the literature involving pairs of matrices that can be simultaneously positive diagonally scaled to orthogonal matrices. As a result, we give below a complete resolution to the 1×1 and 2×2 cases. In so doing the conservation of E-energy problem is solved for the cases where A in (1.1) is of order 4 or less. (In most applications, such as acoustics, the matrix A in (1.1) is indeed of order 4 or less.)

The case where S_1 and S_2 are 1×1 is especially simple. There, if $S_1 = (s_1)$ and $S_2 = (s_2)$, then S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices $Q = (\pm 1)$ if and only if

$$(2.6) \qquad s_1 \neq 0 \quad \text{and} \quad s_2 = \pm 1/s_1.$$

Note that with s_1 and s_2 satisfying (2.6), the positive diagonal matrices $C = (c_1)$ and $C_2 = (c_2)$ must satisfy

$$(2.7) \qquad c_1 c_2 = 1/|s_1|$$

but are otherwise arbitrary. Choosing C_1 and C_2 to satisfy (2.7), the matrix E for which the E-energy is conserved can then be constructed by using (2.4).

The 2×2 case is settled by the following theorem:

THEOREM 4. *Real 2×2 matrices S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices if and only if they have the representations*

$$(2.8) \qquad S_1 = \begin{pmatrix} ac \cdot \cos \theta_1 & \pm ad \cdot \sin \theta_1 \\ bc \cdot \sin \theta_1 & \mp bd \cdot \cos \theta_1 \end{pmatrix},$$

$$(2.9) \qquad S_2 = \begin{pmatrix} \dfrac{\cos \theta_2}{ac} & \pm \dfrac{\sin \theta_2}{bc} \\ \dfrac{\sin \theta_2}{ad} & \mp \dfrac{\cos \theta_2}{bd} \end{pmatrix}$$

for some scalars θ_1, θ_2, and $a, b, c, d > 0$.

Proof. It is known (see Cullen [1, p. 103]) that every 2×2 orthogonal matrix Q has the form

$$(2.10) \qquad Q(\theta) = \begin{pmatrix} \cos \theta & \pm \sin \theta \\ \sin \theta & \mp \cos \theta \end{pmatrix}.$$

Thus, for positive diagonal matrices

$$(2.11) \qquad C_1^{-1} = \operatorname{diag}\{a, b\} \quad \text{and} \quad C_2^{-1} = \operatorname{diag}\{c, d\},$$

S_1 and S_2 have the form

$$S_1 = C_1^{-1} Q(\theta_1) C_2^{-1} \quad \text{and} \quad S_2 = C_2 Q(\theta_2) C_1$$

for some θ_1 and θ_2. The representations (2.8) and (2.9) follow. $\square$

Remark. The 3×3 case can be similarly handled since every 3×3 orthogonal matrix has a representation in terms of two angles analogous to the 2×2 representation (2.10). $\square$

In the next section we present some illustrative examples. Before doing so, we briefly discuss the case of the coefficient matrix A having repeated eigenvalues. Due to the ordering of the eigenvalues of A preceding the partitioning (1.4), repeated eigenvalues are clustered together in Λ. Then (1.10) implies that D is a block diagonal matrix. The diagonal blocks are all to be symmetric and positive definite so that E is likewise symmetric and positive definite. The size and position of the blocks are determined in an obvious manner from the multiplicity of the eigenvalues of A. For example, if $\Lambda = \operatorname{diag}\{3, 2, 2, 1, -1, -1, -2\}$, then D is allowed to have the structure

$$(2.12)$$

where the numbers indicate the block sizes. The lemmas, theorems and corollaries presented above are all still valid with the proviso that positive diagonal matrices are now thought of as positive definite block diagonal matrices. The validity of these results is a direct consequence of the fact that the matrices D and Λ continue to commute. We emphasize that the block diagonal structure exemplified by (2.12) is the most general structure that D may possess if the E-energy is to be conserved. On the other hand, in specific situations, the matrix D may be of simpler form and still the E-energy would be conserved. (See the remark following Corollary 2.2.)

III. Examples. The first example, though admittedly simple, illustrates many of the points discussed in the previous section. Let

$$(3.1) \qquad A = \begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $0 < \mu < 1$ and let the boundary conditions be given by

$$(3.2) \qquad u_1(t, 0) = u_1(t, 1) = 0.$$

Obviously, A is of even order and the eigenvalues of A are given by

$$\lambda_1 = \mu + 1 > 0 \quad \text{and} \quad \lambda_2 = \mu - 1 < 0.$$

The diagonalizing matrix for A may be taken to be

$$(3.3) \qquad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = T^{-1} = T^t,$$

so that $w = Tu$ and the boundary conditions (3.2) yield

$$(3.4) \qquad w_2(t, 0) = - w_1(t, 0) \quad \text{and} \quad w_1(t, 1) = - w_2(t, 1).$$

Therefore, $S_1 = S_2 = (-1)$ so that (2.6) is satisfied. Then, according to (2.7) we may choose $C_1 = C_2 = (1)$. Since $\Lambda_1 = (1 + \mu)$ and $\Lambda_2 = (1 - \mu)$, (2.3) yields that

$$D_1 = 1/(1 + \mu) \quad \text{and} \quad D_2 = 1/(1 - \mu).$$

Then, with T given by (3.3), (2.4) yields that

$$E = \frac{1}{(1 - \mu^2)} \begin{pmatrix} 1 & -\mu \\ -\mu & 1 \end{pmatrix}.$$

We now check to see if indeed E satisfies all our requirements. It is obvious that E is symmetric and since its eigenvalues are $1 \pm \mu$, E is positive definite. Furthermore,

$$EA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that E symmetrizes A. Finally, the E-energy is conserved since from (1.9) and (3.2) we see that

$$dE/dt = \tfrac{1}{2}\{u_1(t, 1)u_2(t, 1) - u_1(t, 0)u_2(t, 0)\} = 0.$$

If the boundary conditions (3.4) are replaced by the conditions $w_2(t, 0) = \nu w_1(t, 0)$ and $w_1(t, 1) = \sigma w_2(t, 1)$, then $|\nu\sigma| = 1$ is necessary and sufficient for the conservation of E-energy. Furthermore, the scalars D_1 and D_2 must be chosen so that

$$D_1\Lambda_1 = \nu^2 D_2\Lambda_2.$$

This simple example is of special interest since its E-energy can be related to a physical energy. With A given by (3.1) the system (1.1) is an equivalent to Galilean transformation of the wave equation, and the E-energy of the solution of (1.1) is equivalent to the physical energy measured in the transformed space-time coordinates (see Gunzburger [4] for details).

Less trivial examples may be built by choosing boundary matrices of the form (2.8) and (2.9) and also choosing a matrix Λ. Then coefficient matrices A may be constructed from (1.2) by choosing any nonsingular matrix T. Furthermore, using (2.3), (2.4) and (2.11), the E matrix may be constructed. For illustration purposes, let the parameters in (2.8) and (2.9) be given by $\theta_1 = \pi/2, \theta_2 = \pi/3, a = d = 1$, and $b = c = 2$. Then

$$S_1 = \begin{pmatrix} 0 & \pm 1 \\ 4 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1/4 & \pm\sqrt{3/8} \\ \sqrt{3}/2 & \pm 1/4 \end{pmatrix}$$

are boundary matrices such that S_1 and S_2^{-1} can be simultaneously positive diagonally scaled to orthogonal matrices. In fact, from (2.11) we see that if $C_1^{-1} = \text{diag}\{1, 2\}$ and $C_2^{-1} = \text{diag}\{2, 1\}$, then

$$C_1 S_1 C_2 = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C_1 S_2^{-1} C_2 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & \mp 1 \end{pmatrix}$$

are indeed orthogonal. If $\Lambda = \text{diag}\{\lambda_1, \lambda_2, -\lambda_3, -\lambda_4\}$, where all the λ_j's are positive, then from (2.3) we may choose

$$D = \text{diag}\{1/\lambda_1, 1/4\lambda_2, 4/\lambda_3, 1/\lambda_4\}.$$

Then, from (1.12), it can easily be verified that the E-energy is conserved.

As further illustrations we list some zero patterns of matrices S_1 and S_2 satisfying Theorem 4. Matrices

$$S_1 = \begin{pmatrix} s_{11} & 0 \\ 0 & s_{22} \end{pmatrix}, \quad S_2 = \begin{pmatrix} t_{11} & 0 \\ 0_1 & t_{22} \end{pmatrix}$$

satisfy Theorem 4 if and only if $s_{11}s_{22} \neq 0$ and $t_{11} = \pm 1/s_{11}$ and $t_{22} = \pm 1/s_{22}$. Matrices

$$S_1 = \begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix}$$

satisfy Theorem 4 if and only if $s_{12}s_{21} \neq 0$ and $t_{12} = \pm 1/s_{21}$ and $t_{21} = \pm 1/s_{12}$. Moreover, combining these two situations, it can be shown more generally that

$$S_1 = \begin{pmatrix} s_{11} & 0 \\ 0 & s_{22} \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix}$$

satisfy Theorem 4 if and only if

(3.5) $$s_{11}s_{22}t_{12}t_{21} = \pm 1.$$

For by Theorem 3, (3.5) is necessary and a simple analysis shows that if (3.5) holds then S_1 and S_2 satisfy (2.8) and (2.9), respectively, with say $\theta_1 = k\pi$, k even if $s_{11} > 0$, k odd if $s_{11} < 0$, with $\theta_2 = (2k + 1)\pi/2$, k even if $t_{21} > 0$, k odd if $t_{21} < 0$, and with say $a > 0$ arbitrary and

$$b = a/|s_{11}| \, |t_{21}|, \quad c = |s_{11}|/a, \quad d = 1/a|t_{12}|.$$

Other pattern pairs can be characterized in a similar manner.

We conclude with the remark that if A is symmetric then the l_2-energy (u, u),

$$\int_0^1 u^t u \, dx,$$

is conserved if and only if $E = D = I$, $S_1^t \Lambda_1 S_1 = \Lambda_2$ and $S_2^t \Lambda_2 S_2 = \Lambda_1$.

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