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On Faster Convergence of the Bisection Method for Certain Triangles

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Abstract. Let ΔABC be a triangle with vertices A, B and C. It is "bisected" as follows: choose a/the longest side (say AB) of ΔABC, let D be the midpoint of AB, then replace ΔABC by two triangles, ΔADC and ΔDBC.

Let Δ_{01} be a given triangle. Bisect Δ_{01} into two triangles Δ_{11}, Δ_{12}. Next, bisect each Δ_{1i}, $i = 1, 2$, forming four new triangles Δ_{2i}, $i = 1, 2, 3, 4$. Continue thus, forming an infinite sequence T_j, $j = 0, 1, 2, \ldots$, of sets of triangles, where $T_j = \{\Delta_{ji}\colon\ 1 \leqslant i \leqslant 2^j\}$. It is known that the mesh of T_j tends to zero as $j \to \infty$. It is shown here that if Δ_{01} satisfies any of four certain properties, the rate of convergence of the mesh to zero is much faster than that predicted by the general case.

1. **Introduction.** Let ΔABC be a triangle with vertices A, B and C. We define the procedure for "bisecting" ΔABC as follows: choose a/the longest side (say AB) of ΔABC, let D be the midpoint of AB, then divide ΔABC into the two triangles ΔADC and ΔDBC.

Let Δ_{01} be a given triangle. Bisect Δ_{01} into two triangles Δ_{11} and Δ_{12}. Next, bisect each Δ_{1i}, $i = 1, 2$, forming four new triangles Δ_{2i}, $i = 1, 2, 3, 4$. Set $T_j = \{\Delta_{ji}\colon\ 1 \leqslant i \leqslant 2^j\}$, $j = 0, 1, 2, \ldots$, so T_j is a set of 2^j triangles. Define m_j, the mesh of T_j, to be the length of the longest side among the sides of the triangles in T_j. Clearly $0 < m_{j+1} \leqslant m_j$ for all $j \geqslant 0$. It is shown implicitly in [3] that in fact $m_j \longrightarrow 0$ as $j \longrightarrow \infty$. Thus, this bisection method is useful in finite element methods for approximating solutions of differential equations (see e.g. [1]). A modification of such a bisection method can be used in computing the topological degree of a mapping from R^3 to R^3 ([4], [5]).

In [2] an explicit bound is obtained for the rate of convergence of m_j: $m_j \leqslant (\sqrt{3}/2)^{[j/2]} m_0$, where $[x]$ denotes the integer part of x. In [2] it is also mentioned that computer experiments indicate that in many cases this bound is unrealistically high; this prompted the present results. We show that if Δ_{01} belongs to any one of four sets of equivalence classes of triangles, then we have the substantially improved bounds of Corollaries 1 and 2 below. Much of the notation used is taken from [3].

2. **Results.**

Definition. Given three positive numbers ρ, σ, τ such that $\rho + \sigma + \tau = \pi$, define (ρ, σ, τ) to be the set of all triangles whose interior angles are ρ, σ, τ.

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With this notation we divide the set of all triangles into similarity classes. Note that $(\rho, \sigma, \tau) = (\sigma, \rho, \tau)$ etc.

Definition. Given a triangle Δ, define $d(\Delta)$, the diameter of Δ, to be the length of the longest side of Δ.

Definition. Given a similarity class (ρ, σ, τ) choose any triangle $\Delta \in (\rho, \sigma, \tau)$ and join the midpoint of its longest side to the opposite vertex. The new side ratio r of (ρ, σ, τ) is defined to be the length of this new side divided by $d(\Delta)$.

Remark. The new side ratio is well-defined since it does not depend on the particular Δ chosen in (ρ, σ, τ). By Lemma 5.2(i) of [4] we have $0 < r \leqslant \sqrt{3}/2$ always.

Definition. Using the notation of the introduction, an iteration of the bisection method applied to Δ_{00} is defined to be the progression from T_j to T_{j+1} for any $j \geqslant 0$. A cycle of the bisection method is defined to be two successive iterations, i.e., the progression from T_j to T_{j+2} for any $j \geqslant 0$.

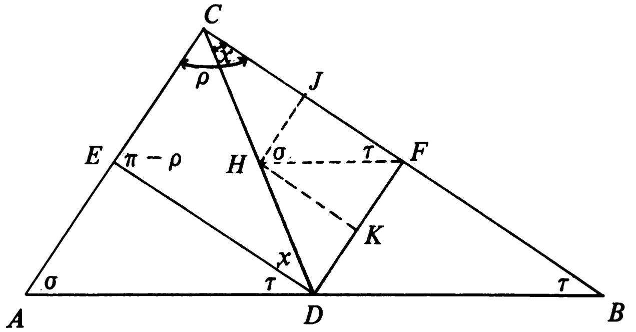

FIGURE 1

In Figure 1, $\Delta ABC \in (\rho, \sigma, \tau)$ and we have taken $\tau \leqslant \sigma \leqslant \rho$. Thus, $AC \leqslant BC \leqslant AB$ (where 'AC' denotes 'length of AC' etc.). D, E, F are the midpoints of AB, AC, BC, respectively. Under the additional hypothesis that $x + \tau \geqslant \max\{\sigma, \rho - x\}$ and $\pi - \rho \geqslant \rho - x$, bisections will take place exactly as in Figure 1 (see [3]).

Notation. To indicate that $\Delta_1 \in (\alpha_1, \beta_1, \gamma_1)$ yields $\Delta_2 \in (\alpha_2, \beta_2, \gamma_2)$ and $\Delta_3 \in (\alpha_3, \beta_3, \gamma_3)$ when bisected, we shall draw a pair of arrows emanating from the triple $(\alpha_1, \beta_1, \gamma_1)$ with one entering each of the triples $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$.

We quote the following result from [3]; it uses the notation of Figure 1.

LEMMA 1 [3, LEMMA 4]. *If* $\tau \leqslant \sigma \leqslant \rho$, $x + \tau \geqslant \max\{\sigma, \rho - x\}$, *and* $\pi - \rho \geqslant \rho - x$, *then we must have*

$$\begin{tikzcd}
(\rho, \sigma, \tau) \arrow[r, shift left] \arrow[d, shift right] & (x, \tau, \rho + \sigma - x) \arrow[l, shift left] \arrow[d, shift right] \\
(\rho - x, \sigma, x + \tau) \arrow[r, shift left] \arrow[u, shift right] & (x, \rho - x, \pi - \rho) \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Proof. See [3].

Thus, after one cycle of the bisection method applied to $\Delta ABC \in (\rho, \sigma, \tau)$, we have two triangles from (ρ, σ, τ) (ΔADE, ΔDBF above) and two triangles from $(x, \rho - x, \pi - \rho)$ (ΔCED, ΔCFD above). Now $d(ADE) = d(DBF) = d(ABC)/2$. Also,

$d(CED) = d(CFD) = CD$ (since $\pi - \rho \geqslant \rho - x$ by hypothesis, and $\rho - x \geqslant x$ from [3]), i.e., $d(CED) = d(CFD) = rd(ABC)$, where r is the new side ratio of (ρ, σ, τ).

Similarly, after one cycle of the bisection method applied to $\Delta CFD \in (x, \rho - x, \pi - \rho)$, we obtain ΔHJF, $\Delta HKF \in (\rho, \sigma, \tau)$ and ΔCJH, $\Delta HKD \in (x, \rho - x, \pi - \rho)$. Here

$$d(CJH) = d(HKD) = d(CFD)/2,$$

and

$$d(HJF) = d(HKF) = HF = AB/4 = CD/4r = d(CFD)/4r.$$

THEOREM 1. *Assume that* $\tau \leqslant \sigma \leqslant \rho, x + \tau \geqslant \max\{\sigma, \rho - x\}$, *and* $\pi - \rho \geqslant \rho - x$. *Then for* $n \geqslant 1$, *after* n *cycles of the bisection method applied to*

(i) $\Delta \in (\rho, \sigma, \tau)$, *we have* 2^{2n-1} *triangles in* (ρ, σ, τ) *each with diameter* $d(\Delta)/2^n$ *and* 2^{2n-1} *triangles in* $(x, \rho - x, \pi - \rho)$ *each with diameter* $d(\Delta)r/2^{n-1}$;

(ii) $\Delta' \in (x, \rho - x, \pi - \rho)$, *we have* 2^{2n-1} *triangles in* $(x, \rho - x, \pi - \rho)$ *each with diameter* $d(\Delta')/2^n$ *and* 2^{2n-1} *triangles in* (ρ, σ, τ) *each with diameter* $d(\Delta')/2^{n+1}r$.

Here r *is the new side ratio of* (ρ, σ, τ).

Proof. We use induction on n. The case $n = 1$ is proven in the remarks following Lemma 1.

Fix $k > 1$. Assume that the theorem is true for $1 \leqslant n < k$. We prove it true for $n = k$.

First, part (i). After one cycle of the bisection method applied to Δ, we have $\Delta_1, \Delta_2 \in (\rho, \sigma, \tau)$ with $d(\Delta_1) = d(\Delta_2) = d(\Delta)/2$, and $\Delta_3, \Delta_4 \in (x, \rho - x, \pi - \rho)$ with $d(\Delta_3) = d(\Delta_4) = rd(\Delta)$. Applying a further $k - 1$ cycles to each of these four triangles, we obtain from Δ_1 and Δ_2 by the inductive hypothesis 2^{2k-2} triangles in (ρ, σ, τ) each with diameter $d(\Delta_1)/2^{k-1} = d(\Delta)/2^k$, and 2^{2k-2} triangles in $(x, \rho - x, \pi - \rho)$ each with diameter $rd(\Delta_1)/2^{k-2} = rd(\Delta)/2^{k-1}$; from Δ_3 to Δ_4 we get 2^{2k-2} triangles in $(x, \rho - x, \pi - \rho)$ each with diameter $d(\Delta_3)/2^{k-1} = rd(\Delta)/2^{k-1}$ and 2^{2k-2} triangles in (ρ, σ, τ) each with diameter $d(\Delta_3)/r2^k = d(\Delta)/2^k$. Adding totals of identical triangles shows that (i) holds for $n = k$.

By an analogous argument (ii) holds for $n = k$. This completes the proof.

COROLLARY 1. *Suppose* $\tau \leqslant \sigma \leqslant \rho, x + \tau \geqslant \max\{\sigma, \rho - x\}$, *and* $\pi - \rho \geqslant \rho - x$. *Then in the notation of the introduction*

(i) *If* $\Delta_{01} \in (\rho, \sigma, \tau)$, *then* $m_j \leqslant \max\{r, ½\}(½)^{[j/2]-1}d(\Delta_{01})$ *for* $j \geqslant 1$, *with equality for even* j;

(ii) *if* $\Delta_{01} \in (x, \rho - x, \pi - \rho)$, *then* $m_j \leqslant \max\{1/2r, 1\}(½)^{[j/2]}d(\Delta_{01})$ *for* $j \geqslant 1$, *with equality for even* j.

Proof. Immediate from Theorem 1.

Remark. In practice the conditions of Theorem 1 are more easily checked if expressed in terms of the lengths of sides of triangles. Using the notation of Figure 1,

$\tau \leqslant \sigma \leqslant \rho$ is equivalent to $AC \leqslant BC \leqslant AB$,

$x + \tau \geqslant \max\{\sigma, \rho - x\}$ is equivalent to $AC \geqslant \max\{AB/2, CD\}$,

$\pi - \rho \geqslant \rho - x$ is equivalent to $CD \geqslant BC/2$.

Thus, knowing the lengths of AC, BC, AB and CD one can immediately decide whether or not $\Delta ABC \in (\rho, \sigma, \tau)$ satisfies the conditions of Theorem 1. Note that these inequalities and [4, Lemma 5.2(i)] give $1/4 \leqslant r \leqslant \sqrt{3}/2$.

Given a triangle such as ΔCFD with $CD \geqslant CF \geqslant DF$, to decide whether or not $\Delta CFD \in (x, \rho - x, \pi - \rho)$ for some (ρ, σ, τ) where the various angles satisfy the conditions of Theorem 1, bisect CD at H and CF at J, then check (as above for ΔABC) whether or not $\Delta HJF \in (\rho, \sigma, \tau)$ satsifies the condtions of Theorem 1 with $HF \geqslant FJ \geqslant JH$.

We now give a theorem similar to Theorem 1 which deals with the other two similarity classes mentioned in Lemma 1.

Definition. The smaller sides ratio s of a similarity class (ρ, σ, τ) is obtained by choosing any $\Delta ABC \in (\rho, \sigma, \tau)$ with $AB \geqslant BC \geqslant AC$, then setting $s = BC/AC$.

THEOREM 2. *Assume that* $\tau \leqslant \sigma \leqslant \rho, x + \tau \geqslant \max\{\sigma, \rho - x\}$, *and* $\pi - \rho \geqslant \rho - x$. *Then for* $n \geqslant 1$, *after* n *cycles of the bisection method applied to*

(i) $\Delta \in (\rho - x, \sigma, x + \tau)$, *we have* 2^{2n-1} *triangles in* $(\rho - x, \sigma, x + \tau)$ *each with diameter* $d(\Delta)/2^n$ *and* 2^{2n-1} *triangles in* $(x, \tau, \rho + \sigma - x)$ *each with diameter* $sd(\Delta)/2^n$:

(ii) $\Delta' \in (x, \tau, \rho + \sigma - x)$, *we have* 2^{2n-1} *triangles in* $(x, \tau, \rho + \sigma - x)$ *each with diameter* $d(\Delta')/2^n$ *and* 2^{2n-1} *triangles in* $(\rho - x, \sigma, x + \tau)$ *each with diameter* $d(\Delta')/2^n s$.

Here s *is the smaller sides ratio of* (ρ, σ, τ).

Proof. Analogous to that of Theorem 1.

COROLLARY 2. *Suppose* $\tau \leqslant \sigma \leqslant \rho, x + \tau \geqslant \max\{\sigma, \rho - x\}$, *and* $\pi - \rho \geqslant \rho - x$. *Then in the notation of the introduction*

(i) *if* $\Delta_{01} \in (\rho - x, \sigma, x + \tau)$, *then* $m_j \leqslant s(\frac{1}{2})^{[j/2]} d(\Delta_{01})$ *for* $j \geqslant 1$, *with equality for even* j;

(ii) *if* $\Delta_{01} \in (x, \tau, \rho + \sigma - x)$, *then* $m_j \leqslant (\frac{1}{2})^{[j/2]} d(\Delta_{01})$ *for* $j \geqslant 1$, *with equality for even* j.

Note that since we are assuming that $AC \geqslant \max\{AB/2, CD\}$ in Figure 1, we have $1 \leqslant s \leqslant 2$.

Remark. Given a triangle ΔRST with $RS = d(RST)$, to decide whether or not $\Delta RST \in (\rho - x, \sigma, x + \tau)$ or $(x, \tau, \rho + \sigma - x)$ for some (ρ, σ, τ) where the various angles satisfy the conditions of Theorem 2, bisect RS at W (say). Examine the triangles ΔRWT and ΔWST. If one of these is in (ρ, σ, τ), where the angles satisfy the conditions of Theorem 2, then

(i) $2WT \geqslant RS \Rightarrow \Delta RST$ is in the corresponding $(\rho - x, \sigma, x + \tau)$,

(ii) $2WT \leqslant RS \Rightarrow \Delta RST$ is in the corresponding $(x, \tau, \rho + \sigma - x)$.

Various conditions sufficient for a given triangle ΔXYZ to lie in one of the four sets of similarity classes considered can be obtained by elementary calculations using the cosine rule for triangles. For example, given $\Delta XYZ \in (\alpha, \beta, \gamma)$ with $XY \geqslant YZ \geqslant XZ$ and $\alpha \geqslant \beta \geqslant \gamma$, then

(i) if $\cos \gamma \leqslant 3/4$, then $\Delta XYZ \in (\rho, \sigma, \tau)$ satisfies the conditions of Theorem 1;

(ii) if $XY/YZ \geqslant 2/\sqrt{3}$ and $\cos \gamma \leqslant \sqrt{3}/2$, then $\Delta XYZ \in (\rho, \sigma, \tau)$ and $\Delta XYZ \in (x, \rho - x, \pi - \rho)$, both satisfying the conditions of Theorem 1;

(iii) if $3/4 \geqslant \cos \beta \geqslant \max\{XZ/XY, XY/4XZ\}$, then $\Delta XYZ \in (x, \tau, \rho + \sigma - x)$ satisfies the conditions of Theorem 2.

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1. G. DAHLQUIST & A. BJORCK, *Numerical Methods*, translated by N. Anderson, Prentice-Hall, Englewood Cliffs, N. J., 1974.

2. R. B. KEARFOTT, "A proof of convergence and an error bound for the method of bisection in R^n," *Math. Comp.*, v. 32, 1978, pp. 1147–1153.

3. I. G. ROSENBERG & F. STENGER, "A lower bound on the angles of triangles constructed by bisecting the longest side," *Math. Comp.*, v. 29, 1975, pp. 390–395.

4. M. STYNES, "An algorithm for numerical calculation of topological degree," *Applicable Anal.* (To appear.)

5. M. STYNES, "On the construction of sufficient refinements for computation of topological degree," *Numer. Math.* (Submitted.)