On the Preceding Paper
“A Legendre Polynomial Integral”
by James L. Blue

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Abstract. The modified moments of the distribution \( d\sigma(x) = x^\alpha \ln(1/x) \, dx \) on \([0, 1]\), with respect to the shifted Legendre polynomials, are explicitly evaluated.

The result in the theorem of Section 2 of [1] can be generalized as follows: Let
\[
\nu_n(\alpha) = \int_0^1 x^\alpha \ln(1/x) P_n^*(x) \, dx, \quad \alpha > -1, \quad n = 0, 1, 2, \ldots ,
\]
where \( P_n^*(x) = P_n(2x - 1) \) is the shifted Legendre polynomial of degree \( n \). Then

\[
\nu_n(\alpha) = \begin{cases} 
(-1)^{n-m} \frac{m!^2 (n-m-1)!}{(n+m+1)!}, & \alpha = m < n, m \geq 0 \text{ an integer,} \\
\frac{1}{\alpha+1} \left( \frac{1}{\alpha+1} + \sum_{k=1}^{n} \left( \frac{1}{\alpha+1+k} - \frac{1}{\alpha+1-k} \right) \right) \frac{\alpha+1-k}{\alpha+1+k}, & \text{otherwise.}
\end{cases}
\]

The result in [1] is the case \( \alpha = 0 \) of (1). For the proof, we note that

\[
\nu_n(\alpha) = -2^{-(\alpha+1)} \int_{-1}^{1} (1+t)^\alpha \ln (\frac{1}{2}(1+t)) P_n(t) \, dt
\]

\[
= -2^{-(\alpha+1)} \lim_{\nu \to \infty} \left\{ \int_{-1}^{1} (1+t)^\alpha \ln (1+t) P_\nu(t) \, dt - \ln 2 \cdot \int_{-1}^{1} (1+t)^\alpha P_\nu(t) \, dt \right\},
\]

where \( P_\nu(t) \) is the Legendre function of degree \( \nu \). It is well known [2, p. 316, Eq. (15)] that

\[
\int_{-1}^{1} (1+t)^\alpha P_\nu(t) \, dt = \frac{2^{\alpha+1} \Gamma^2(\alpha+1)}{\Gamma(\alpha+\nu+2) \Gamma(\alpha+1-\nu)}, \quad \alpha > -1.
\]

Differentiating (3) with respect to \( \alpha \) gives

\[
\int_{-1}^{1} (1+t)^\alpha \ln (1+t) P_\nu(t) \, dt
\]

\[
= \frac{2^{\alpha+1} \Gamma^2(\alpha+1)}{\Gamma(\alpha+\nu+2) \Gamma(\alpha+1-\nu)} \left\{ \ln 2 + 2\psi(\alpha+1) - \psi(\alpha+\nu+2) - \psi(\alpha+1-\nu) \right\},
\]

with \( \psi(x) = \Gamma'(x)/\Gamma(x) \) the logarithmic derivative of the gamma function. The assertion (1) now follows by inserting (3) and (4) in (2) and by using the recurrence relations
\( \Gamma(x+1) = x\Gamma(x), \psi(x+1) = \psi(x) + 1/x \), together with the fact that for any integer

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\[ r \geq 0, \]
\[ \frac{\psi(-r + \varepsilon)}{\Gamma(-r + \varepsilon)} \to (-1)^{r-1}r! \quad \text{as } \varepsilon \to 0. \]

The method of proof also allows the evaluation of integrals of the form
\[ \nu_{n,k}(\alpha) = \int_0^1 x^\alpha \ln(1/x)^k P_n^*(x) \, dx, \]
by repeatedly differentiating (4) with respect to \( \alpha \).

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