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On the Preceding Paper "A Legendre Polynomial Integral" by James L. Blue

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Abstract. The modified moments of the distribution $d\sigma(x) = x^\alpha \ln(1/x)\,dx$ on $[0, 1]$, with respect to the shifted Legendre polynomials, are explicitly evaluated.

The result in the theorem of Section 2 of [1] can be generalized as follows: Let

$$\nu_n(\alpha) = \int_0^1 x^\alpha \ln(1/x) P_n^*(x)\,dx, \qquad \alpha > -1, \quad n = 0, 1, 2, \ldots,$$

where $P_n^*(x) = P_n(2x - 1)$ is the shifted Legendre polynomial of degree n. Then

$$(1)\quad \nu_n(\alpha) = \begin{cases} (-1)^{n-m} \dfrac{m!^2(n-m-1)!}{(n+m+1)!}, & \alpha = m < n,\ m \geqslant 0 \text{ an integer}, \\[2ex] \dfrac{1}{\alpha+1}\left\{\dfrac{1}{\alpha+1} + \displaystyle\sum_{k=1}^n \left(\dfrac{1}{\alpha+1+k} - \dfrac{1}{\alpha+1-k}\right)\right\} \displaystyle\prod_{k=1}^n \dfrac{\alpha+1-k}{\alpha+1+k}, & \text{otherwise.} \end{cases}$$

The result in [1] is the case $\alpha = 0$ of (1). For the proof, we note that

$$(2)\quad \begin{aligned} \nu_n(\alpha) &= -2^{-(\alpha+1)} \int_{-1}^1 (1+t)^\alpha \ln(\tfrac12(1+t)) P_n(t)\,dt \\ &= -2^{-(\alpha+1)} \lim_{\nu\to n} \left\{\int_{-1}^1 (1+t)^\alpha \ln(1+t) P_\nu(t)\,dt - \ln 2 \cdot \int_{-1}^1 (1+t)^\alpha P_\nu(t)\,dt\right\}, \end{aligned}$$

where $P_\nu(t)$ is the Legendre function of degree ν. It is well known [2, p. 316, Eq. (15)] that

$$(3)\qquad \int_{-1}^1 (1+t)^\alpha P_\nu(t)\,dt = \frac{2^{\alpha+1}\Gamma^2(\alpha+1)}{\Gamma(\alpha+\nu+2)\Gamma(\alpha+1-\nu)}, \qquad \alpha > -1.$$

Differentiating (3) with respect to α gives

$$(4)\quad \begin{aligned} &\int_{-1}^1 (1+t)^\alpha \ln(1+t) P_\nu(t)\,dt \\ &\quad = \frac{2^{\alpha+1}\Gamma^2(\alpha+1)}{\Gamma(\alpha+\nu+2)\Gamma(\alpha+1-\nu)} \{\ln 2 + 2\psi(\alpha+1) - \psi(\alpha+\nu+2) - \psi(\alpha+1-\nu)\}, \end{aligned}$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$ the logarithmic derivative of the gamma function. The assertion (1) now follows by inserting (3) and (4) in (2) and by using the recurrence relations $\Gamma(x+1) = x\Gamma(x)$, $\psi(x+1) = \psi(x) + 1/x$, together with the fact that for any integer

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$r \geqslant 0$,

$$\frac{\psi(-r+\epsilon)}{\Gamma(-r+\epsilon)} \longrightarrow (-1)^{r-1} r! \quad \text{as } \epsilon \longrightarrow 0.$$

The method of proof also allows the evaluation of integrals of the form

$$\nu_{n,k}(\alpha) = \int_0^1 x^\alpha [\ln (1/x)]^k P_n^*(x)\, dx,$$

by repeatedly differentiating (4) with respect to α.

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1. J. L. BLUE, "A Legendre polynomial integral," *Math. Comp.*, v. 33, 1979, pp. 739–741.
2. A. ERDÉLYI (Ed.), *Tables of Integral Transforms*, Vol. II, McGraw-Hill, New York, 1954.