The Hankel Power Sum Matrix Inverse and the Bernoulli Continued Fraction

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Abstract. The $m \times m$ Hankel power sum matrix $W = V V^T$ (where $V$ is the $m \times n$ Vandermonde matrix) has $(i, j)$-entry $S_{i+j-2}(n)$, where $S_p(n) = \sum_{k=1}^{n} k^p$. In solving a statistical problem on curve fitting it was required to determine $f(m)$ so that for $n > f(m)$ all eigenvalues of $W^{-1}$ would be less than 1. It is proved, after calculating $W^{-1}$ by first factoring $W$ into easily invertible factors, that $f(m) = (13m^2 - 5)/8$ suffices. As by-products of the proof, close approximations are given for the Hilbert determinant, and a convergent continued fraction with $m$th partial denominator $m^{-1} + (m + 1)^{-1}$ is found for the divergent Bernoulli number series $\sum B_{2k}(2x)^{2k}$.

1. Introduction. Defined as the product $W = V V^T$ of the $m \times n$ Vandermonde matrix $V = (j^{i-1})$ with its transpose $V^T$, the $m \times m$ Hankel power sum matrix $W = W_m$ has $(i, j)$-entry $S_{i+j-2}(n)$, where

$$S_p = S_p(n) = \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{B_{2k}}{2k} \left( \frac{p}{2k} - 1 \right),$$

and where $B_{2k}$ are the Bernoulli numbers [3], [6]:

$$B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots .$$

In solving a statistical problem involving the fitting of polynomial curves of degree $m$ to $n > m$ points, for increasing $m$ and $n$, it was required [4] to find a function $f(m)$, such that, whenever $n > f(m)$, the eigenvalues $\mu_k$ of $M = W^{-1}$ would all be less than 1. We evaluate $w_m = \det W_m$ in Section 2 as

$$w_m = \det W_m = h_m \prod_{i,j=1}^{m} (n + i - j),$$

where $h_m$ is the determinant of the Hilbert matrix $H_m$, and we obtain close estimates for $h_m$. In Section 3 we factor $W_m$ as a product of easily invertible matrices of which only diagonal matrices involve $n$, and we also explicitly invert $W_m$ and $H_m$. In Section 4 we estimate the trace of $M = W^{-1}$ and find that the function

$$f(m) = (13m^2 - 5)/8$$

suffices for powers of $M$ to converge when $n > f(m)$.

As a by-product of this investigation, we find in Section 5 that the divergent...
asymptotic series $B(x)$ with Bernoulli number coefficients

$$B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k},$$

related to the Laplace transform of $x \coth x - 1$, has the convergent continued fraction expansion

$$B(x) = \frac{x^2}{|1 + 1/2 +} \frac{x^2}{|1/2 + 1/3 +} \frac{x^2}{|1/3 + 1/4 +} \ldots .$$

In fact, $B(1/12) = \pi^2 - 9.865$ is given with error $< 2 \times 10^{-12}$ by the sixth convergent of this continued fraction.

2. The Determinants. Since $S_0(n) = n$ and $n(n+1)/2$ divides $S_k(n)$ for $k > 0$, it follows directly that $w_m = \det W_m$ has the algebraic factor $n^m(n+1)^{m-1}$. For $r < m$, $(n-r)^{m-r}$ is also an algebraic factor of $w_m$, since the matrix $W_m(n)$ has rank $r$ and nullity $m-r$, when $n = r$ for $r = 1, 2, \ldots, m-1$. Since the polynomials $S_p(n)$ are generated by the function

$$G(x, n) = \frac{e^{xn} - 1}{1 - e^{-x}} = \sum_{p=0}^{\infty} S_p(n)x^p/p!,$$

we see from the identity

$$G(x, n) + G(-x, -n-1) + 1 = 0$$

that

$$S_p(n) + (-1)^p S_p(-n-1) + \delta_{p,0} = 0.$$

Hence, $w_m$ has the algebraic factor $(n+1+r)^{m-1-r}$. We have found $m^2$ linear functions of $n$ as factors of the polynomial $w_m(n)$ which is of degree $m^2$ in $n$. The remaining factor is the determinant of the leading coefficients $1/(i+j-1)$, namely the determinant $h_m$ of the ill-conditioned Hilbert matrix $H_m$ of order $m$ [5], [7]. This proves Eq. (1.3).

If we take $m = n$ in (1.3), the Vandermonde matrix $V$ in $W = VV^T$ is square. Its determinant $v_m$ is

$$v_m = \det V_m = 1! \cdot 2! \cdot 3! \cdot \ldots \cdot (m-1)! \equiv (m-1)!!.$$

Hence by (1.3) and (2.3)

$$h_m = \det V_m^2 = \prod_{i,j=1}^{m} (m + i - j) = v_m^4/v_{2m}.$$ 

The ratio of successive $h_m$'s is

$$h_{m+1}/h_m = (2m+1)!(2m)!/m!^4 = (2m+1) \binom{2m}{m}^2.$$

The following theorem gives a close approximation for $h_m$.

**Theorem 2.1.** The determinant $h_m$ of the Hilbert matrix $H_m$ of order $m$ is
given to 10 significant figures for \( m > 4 \) by

\[
(2.6) \quad h_m = 4^{-m(m-1)}(\pi/2)^{m-1}m^{-1/4}\exp R_m,
\]

where the remainder function \( R_m \) is defined by

\[
(2.7) \quad R_m = \int_0^\infty (e^{-2t} - e^{-2mt}) \tanh^2 (t/2)(4t)^{-1} dt
\]

and is approximated to 9 decimals for \( m \geq 5 \) by

\[
(2.8) \quad R_m = 0.013081539 - 2^{-6}m^{-2} + 2^{-8}m^{-4} - 2^{-8.5}m^{-6} + 2^{-8}m^{-8} - 2^{-7}m^{-10}.
\]

**Proof.** The ratio \( h_m/h_{m+1} \) in (2.5) is related to the Wallis approximation \( \pi_m \) for \( \pi \) by

\[
(2.9) \quad \frac{\pi_m}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \ldots \frac{2m}{2m-1} \frac{2m}{2m+1} = 2^4 m h_{m+1}/h_m.
\]

If we express \( \ln n \) in the form

\[
(2.10) \quad \ln n = \int_1^n \frac{ds}{s} = \int_0^\infty \int_1^n e^{-st} ds \, dt = \int_0^\infty (e^{-t} - e^{-nt})t^{-1} dt,
\]

then \( \ln(\pi_m/2) \) and its limit \( \ln(\pi/2) \) are expressible as

\[
(2.11) \quad \ln(\pi_m/2) = \int_0^\infty (e^{-t} - 2e^{-2t} + 2e^{-3t} - 2e^{-2mt} + e^{-(2m+1)t})t^{-1} dt
\]

\[
= \int_0^\infty (e^{-t} - e^{-(2m+1)t})(1 - e^{-t})(1 + e^{-t})^{-1} t^{-1} dt,
\]

\[
(2.12) \quad \ln(\pi/2) = \int_0^\infty e^{-t} \tanh (t/2)t^{-1} dt,
\]

\[
(2.13) \quad \ln(\pi/\pi_k) = \int_0^\infty e^{-(2k+1)t} \tanh (t/2)t^{-1} dt,
\]

\[
(2.14) \quad (\pi/2)^{m-1} \prod_{k=1}^{m-1} (\pi_k/\pi) = 2^2 m(m-1) h_m / h_1.
\]

Summing in (2.13) from \( k = 1 \) to \( m - 1 \) yields

\[
(2.15) \quad \ln[(\pi/2)^{m-1}2^{-2m(m-1)/h_m}] = \int_0^\infty (e^{-2t} - e^{-2mt})(e^{t/2} + e^{-t/2})^{-2} t^{-1} dt
\]

\[
= (1/4) \ln(2m/2) - R_m
\]

by (2.10), where \( R_m \) is defined by (2.7). Equation (2.6) follows from (2.15). To obtain (2.8) we evaluate \( R_4 = .012119610988 \) from (2.6) setting \( h_4 = 1/6048000 \) in (2.6). Then we compute \( R_\infty - R_m \) from (2.7) by replacing \( \tanh(t/2) \) by the first five terms of its series, and set \( m = 4 \) to get \( R_\infty \) in (2.8). We check the tenth decimal by working from \( h_5 \) instead. This gives \( \exp R_m \) and \( h_m \) accurate to 10 significant figures.
For $m = 20$ we find $R_{20} = 0.130425009$ and

$$h_{20} = 4.206178954 \times 10^{-226}.$$  

The matrices $H_m$ and $W_m(n)$ are ill conditioned. In fact, $W_3(3)$ has the eigenvalues $\lambda_1 = 113.4132$, $\lambda_2 = 1.564253$, $\lambda_3 = 0.02254695$ and the conditioning ratio $\lambda_1/\lambda_3 = 5030$. So the usual computer methods for inverting $W_m(n)$ are unreliable [5], [7].

3. Inversion by Factoring. To invert the ill-conditioned $m \times m$ matrix $W = W_m(n)$ with $(i, j)$-entry $S_{i+j-2}(n)$, we first factor it into easily invertible factors, restricting the variable $n$ to diagonal matrix factors $E_{\pi} = (\text{diag} \ e_{\pi})$ and $Q = (\text{diag} \ q_j)$, where

$$e_i = (-1)^{i-1}, \quad p_i = \binom{n + i - 1}{n - m}, \quad q_j = \binom{n - j}{n - m}.$$  

We denote by $T = (t_{ij})$ the lower Pascal triangle matrix with

$$t_{ij} = \binom{i - 1}{j - 1} = \binom{i - 1}{i - j} = (-1)^{i+j} \binom{-j}{i - j}.$$  

We note that $ETE$ has entries $(-j^i_{i-j})$, so

$$(TETE)_{ij} = \sum_{k=j}^{i} \binom{i - 1}{i - k} \binom{-j}{k - j} = \binom{i - j - 1}{i - j} = \delta_{ij}$$

and $T^{-1} = ETE$. Next, we define an $m \times m$ lower triangular row stochastic matrix $A = (a_{ij})$ that converts the integral powers in $V$ into binomial coefficients by the formula

$$(AV)_{ik} = \sum_{r=1}^{i} a_{ir} k^{r-1} = \binom{k + i - 2}{i - 1}, \quad k = 1, 2, \ldots, n.$$  

The $a_{ir}$ are related to Stirling numbers of the first kind, and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2! & 1/2! & 0 & 0 & \cdots \\ 0 & 2/3! & 3/3! & 1/3! & 0 & \cdots \\ 0 & 6/4! & 11/4! & 6/4! & 1/4! & \cdots \end{bmatrix} = (a_{ij}).$$

From (3.5) and (3.4) we obtain

$$(AV)_{ik} = \binom{i + k - 2}{i - 1} = \sum_{j=1}^{i} \binom{i - 1}{i - j} \binom{k - j}{j - 1} = \sum_{j=1}^{k} t_{ij} \binom{k - 1}{j - 1}.$$  

To factor $W$ we now write
\[
(AV(T^{-1}AV)^T)_{ij} = \sum_{k=j}^{n} \binom{k+i-2}{i-1} \binom{k-1}{j-1}
\]
\[
= \binom{i+j-2}{i-1} \sum_{k=j}^{n} \binom{k+i-2}{j+i-2}.
\]

Summing over \(k\) yields

\[
(AW(T^{-1}A)^T)_{ij} = \binom{i+j-2}{i-1} \binom{n+i-1}{j+i-1}
\]
\[
= \frac{p_i(i+j-2)}{i-1} \binom{m+i-1}{j+i-1}/q_j.
\]

**Theorem 3.1.** The inverse matrix \(M = W^{-1}\) has the factorization

\[
M = W^{-1} = A^T E B A, \quad B = T^T Q T T^T T P^{-1},
\]

where \(E, P, Q, T, A\) are defined in (3.1), (3.2) and (3.5).

**Proof.** Entries of \(TT^T\) and \(TT^T T\) are

\[
(TT^T)_{ik} = \sum_{r=1}^{i} \binom{i-1}{i-r} \binom{k-1}{r-1} = \binom{i+k-2}{i-1},
\]
\[
(TT^T T)_{ij} = \sum_{k=j}^{m} \binom{i+k-2}{k-1} \binom{k-1}{j-1} = \binom{i+j-2}{i-1} \sum_{k=j}^{m} \binom{k+i-2}{j+i-2}.
\]

Summing over \(k\) yields

\[
(TT^T T)_{ij} = \binom{i+j-2}{i-1} \binom{m+i-1}{j+i-1}.
\]

Combining (3.8) and (3.12), we have

\[
AWA^T = PTT^T Q T T^T T P^{-1}.
\]

Since the diagonal sign matrix \(E\) commutes with \(P\) and \(Q\) but transforms \(T\) and \(TT^T\) into their inverses,

\[
(AWA^T)^{-1} = ET^T Q T T^T T P^{-1} E = EBE
\]

and (3.9) is proved.

Equation (3.12) provides a simple method for inverting the Hilbert matrix \(H\).

**Theorem 3.2.** The inverse of the \(m \times m\) Hilbert matrix \(H = (h_{ij})\) with \((i, j)\)-entry \(h_{ij} = 1/(i + j - 1)\) is given by

\[
(H^{-1})_{ij} = d_i d'_i h_{ij} d_j d'_j,
\]
\[
d'_i = \binom{-m-1}{i-1}, \quad d_j = m \binom{m-1}{j-1}.
\]
Proof. Factoring (3.12) yields

\[(3.16) \quad (E^{TT}T)_{ij} = (-1)^{i-1} \binom{m + i - 1}{i - 1} \frac{m}{i + j - 1} \binom{m - 1}{j - 1} = d_i^j h_{ij} d_j.\]

Since $E^{TT}T$ is involutory, $D'HD = (D'HD)^{-1}$. Also,

\[(3.17) \quad h_m = \det H_m = \pm \sqrt[1]{\prod_{i=1}^{m} d_i d_i^T}.\]

Although the matrix $B$ in (3.9) is symmetric, its symmetry is not obvious from formula (3.9).

**Theorem 3.3.** The symmetric matrix $B = (b_{ij})$ in (3.9) has entries expressible in terms of descending factorials $(x)_r = x(x - 1) \cdots (x - r + 1)$ as follows:

\[(3.18) \quad b_{ij} = \sum_{r > i+j-1} \frac{(m + i - 1)_r (m + j - 1)_r}{(n - m + r)!} \binom{r - 1}{i - 1, j - 1},\]

where $\binom{r}{i, j}$ is the trinomial coefficient

\[
\binom{r - 1}{i - 1, j - 1} = \begin{pmatrix}
    r - 1 \\
    i + j - 2
\end{pmatrix}
\begin{pmatrix}
    i + j - 2 \\
    i - 1
\end{pmatrix}.
\]

Proof. To transform $B$ in (3.9) we evaluate

\[(3.19) \quad (T^T Q T T^T)_{is} = \sum_{r=i}^{m} \binom{r - 1}{i - 1} \binom{n - r}{n - m} \binom{r + s - 2}{r - 1},\]

\[
= \binom{i + s - 2}{i - 1} \sum_{r=i}^{m} \binom{r + s - 2}{i + s - 2} \binom{n - r}{n - m},
\]

\[
= \binom{i + s - 2}{i - 1} \binom{n + s - 1}{m - i},
\]

\[(3.20) \quad (B P)_{ij} = \sum_{s=j}^{m} \binom{i + s - 2}{i - 1} \binom{n + s - 1}{m - i} \binom{j - 1}{s - 1},\]

\[
= \binom{i + j - 2}{i - 1} \sum_{s=j}^{m} \binom{i + s - 2}{s - j} \sum_{r=i+j-1}^{m} \binom{s - j}{r - i - j + 1} \binom{n + j - 1}{m + j - 1 - r},
\]

\[
= \sum_{r=i+j-1}^{m} \sum_{s=j}^{m} \binom{i + s - 2}{r - 1} \binom{r - 1}{i - 1, j - 1} \binom{n + j - 1}{n - m + r}.
\]

Summing over $s$ and dividing by $p_j$, we have

\[(3.21) \quad b_{ij} = \sum_{r=i+j-1}^{m} \binom{m + i - 1}{r} \binom{r - 1}{i - 1, j - 1} \binom{m + j - 1}{r} / \binom{n - m + r}{r}.
\]

Writing $(x)_r = r! \binom{x}{r}$, Eq. (3.21) becomes (3.18).
To conserve space in displaying the symmetric matrices $M_m(n) = W_m^{-1}(n)$ we show the upper half of $M_3$ and the lower half of $M_4$.

\[(3.22)\]

\[
\begin{bmatrix}
16\pi^2 + 24n^2 + 56n + 24 \\
\frac{-120(n^2 + n) - 100}{(n)_4} \\
\frac{-120(2n + 1)}{(n)_4} \\
\frac{-140}{(n)_4} \\
\end{bmatrix}
\begin{bmatrix}
9(n^3 + n) + 6 \\
\frac{-18(2n + 1)}{(n)_3} \\
\frac{-180(n + 1)}{(n + 2)_4} \\
\frac{2800}{(n + 3)_4} \\
\end{bmatrix}
\begin{bmatrix}
120(\pi^4 + 4.5\pi^3 + 7\pi^2 + 11\pi + 6) \\
\frac{(24n + 12)(8n + 1)}{(n + 3)_4} \\
\frac{360(2n + 1)(n + 13)}{(n + 2)_3} \\
\frac{-180(n + 1)}{(n + 2)_4} \\
\end{bmatrix}
\begin{bmatrix}
\frac{-300(n^2 + 2n + 2)(3n + 5)}{(n + 3)_4} \\
\frac{-4200(n + 1)}{(n + 2)_5} \\
\frac{360(2n + 1)^{13} + 13}{(n + 3)_4} \\
\frac{180}{(n + 2)_6} \\
\end{bmatrix}
\begin{bmatrix}
\frac{-120(n^2 + n)}{(n)_4} \\
\frac{-300(n + 1)(3n + 2)(3n + 5)}{(n + 3)_3} \\
\frac{2800}{(n + 3)_5} \\
\frac{-4200(n + 1)}{(n + 2)_4} \\
\end{bmatrix}
\]

4. Estimation of $\text{tr} M$. Since $W$ and $M$ are positive definite for $n > m$, all eigenvalues $\mu_k$ of $M$ will satisfy $\mu_k < 1$ if $\text{tr} M < 1$, $m > 1$. For $A$ in (3.5) and $e_i = (-1)^{i-1}$, the first and last diagonal entries of $M = M_m$ are $b_{11}$ and $b_{mm}((m - 1)!^2)$. Numerical computation shows that the maximum $n$ for which $\det(W_m(n) - I) = 0$ are given for $m = 1, 2, 3, 4$ by

\[(4.1)\]

\[(m, n) = (1, 1), (2, 5.82090), (3, 13.3776), (4, 24.24453).\]

The parabola through the first three points is

\[(4.2)\]

\[n = g(m) = 1.5679m^2 - 0.828m - 0.4851,\]

and we find $g(4) = 24.270 > 24.4453$. A slightly higher value than (4.2) will be required for $\text{tr} M < 1$. We first estimate the dominant diagonal entry $b_{11}$ of $M$.

\[(4.3)\]

\[b_{11} = \sum_{r=1}^{m} \left( \begin{array}{c} m \\ r \end{array} \right)^2 \left( \begin{array}{c} n \\ m \end{array} \right)^2 \left( \begin{array}{c} n - m + r \\ r \end{array} \right) = \sum_{r=1}^{m} \left( \begin{array}{c} m \\ r \end{array} \right)^2 \left( \begin{array}{c} n \\ m - r \end{array} \right) = \left( \begin{array}{c} n + m \\ m \end{array} \right) - \left( \begin{array}{c} n \\ m \end{array} \right).\]

\[(4.4)\]

\[1 + b_{11} = \left( \begin{array}{c} n + m \\ m \end{array} \right)^2 \left( \begin{array}{c} n \\ m \end{array} \right) = \prod_{k=1}^{m} \frac{2n + 1 + (2k - 1)}{2n + 1 - (2k - 1)},\]

\[(4.5a)\]

\[\ln(1 + b_{11}) = \sum_{k=1}^{m} \ln \frac{1 + (2k - 1)/(2n + 1)}{1 - (2k - 1)/(2n + 1)} = \sum_{r=1}^{m} \frac{\theta(m, r)}{(2r - 1)(2n + 1)^{2r-1}},\]

where

\[(4.5b)\]

\[\theta(m, r) = \sum_{k=1}^{m} 2(2k - 1)^{2r-1} < \int_{0}^{2m} x^{2r-1} \, dx = (2m)^{2r}/2r.\]

We now assume the inequalities $n > f(m)$ in (1.4).

**Theorem 4.1.** The matrix $M = W_m^{-1}(n)$ has trace $< 1$ if

\[(4.6)\]

\[n > 1.625m^2 - .625 \quad \text{and} \quad m \geq 5.\]
Proof. If (4.6) is satisfied for \( m = 5 \), then \( n > 40 \), and

\[
(4.7a) \quad b_{11} \leq (45)_5/(40)_5 - 1 = 62639/73112 = 0.856754.
\]

If (4.6) is satisfied for \( m > 6 \), then \((2n + 1)/2m^2 > 1.6215\) and

\[
\ln(1 + b_{11}) < \frac{2m^2}{2n + 1} \sum_{r=1}^{\infty} \left( \frac{2m}{2n + 1} \right)^{2r-2}/(2r-1)
\]

\[
< \frac{1}{1.6215} \sum_{r=1}^{\infty} \left( \frac{1}{9.729} \right)^{2r-2}/(2r-1) < 0.62777,
\]

\[
(4.7b) \quad b_{11} < 0.8734 \quad \text{for} \quad m > 6, n > (13m^2 - 5)/8.
\]

The rest of \( \text{tr} M \) is given by

\[
(4.9) \quad \text{tr} M - b_{11} = \sum_{k=2}^{m} \sum_{i,j=k}^{m} a_{ik}(-1)^ib_{ij}(-1)^ja_{jk}.
\]

We replace \( i, j, r \) by \( i + 1, j + 1, r + 2 \) and write

\[
(4.10) \quad b_{i+1,j+1} = \sum_{k=1}^{m-1} y_{ij}^{(r)}, \quad y_{ij}^{(r)} = \frac{(m + j)(m + j - 1)}{(m - m + r + 2)(r + 2)!} \left( \begin{array}{c} r + 1 \\ i, j \end{array} \right) y_{ij}^{(r)}.
\]

Then

\[
(4.11) \quad \text{tr} M - b_{11} = y_{11}^{(1)} \sum_{r=1}^{2m-1} \varphi_{mn}(r), \quad \varphi_{mn}(r) = \sum_{i+j=2}^{r+1} c_{ij}y_{ij}^{(r)}/y_{11}^{(1)},
\]

where the entries of the \((m - 1) \times (m - 1)\) matrix \( C = (c_{ij}) \) are

\[
(4.12a) \quad c_{ij} = (-1)^{i+j}\sum_{k=1}^{m-1} a_{i+1,k}a_{j+1,k} = c_{ij},
\]

\[
(4.12b) \quad C = \frac{1}{720} \begin{bmatrix} 720 & -360 & 240 & -180 & 144 & \cdots \\
-360 & 360 & -300 & 255 & -222 & \cdots \\
240 & -300 & 280 & -255 & 233 & \cdots \\
-180 & 255 & -255 & 242.5 & -228.5 & \cdots \\
144 & -222 & 233 & -228.5 & 220.1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.
\]

The dominant term \( y_{11}^{(1)} \) satisfies

\[
(4.13) \quad y_{11}^{(1)} < \frac{(m + 1)(m + 1)/3}{(3m^2/8 - m + 19/8)_3} \leq \frac{6\delta_{3}/3}{(38)_3} = \frac{200}{2109} < 0.094832,
\]

since the rational function decreases for \( 5 < m \). The function \( \varphi_{mn}(1) \) is 1, but for \( r > 1 \), then \( \varphi_{mn}(r) \) in (4.11) are bounded by rational functions which increase for
$m \geq 5$, and which we replace by their limits as $m \to \infty$.

\[
\varphi_{mn}(2) = (y_{11}^{(2)} - y_{12}^{(2)})/y_{11}^{(1)} = 3(m - 2)(m - 6)/(13m^2 - 8m + 27)
\]
\[< \frac{3}{13} = 0.23077,
\]

\[
\varphi_{mn}(3) = (y_{11}^{(3)} - y_{12}^{(3)} + 2y_{13}^{(3)}/3 + y_{22}^{(3)}/2)/y_{11}^{(1)}
\]
\[< \frac{17(m^2 - 6m + 32)(m - 2)(m - 2.4)/(120)(13m^2/8 - m + 35/8)^2}{(17/120)(8/13)^2} = 0.05365.
\]

Similar calculations yield

\[
\varphi_{mn}(4) < (1/32)(8/13)^3 = 0.00728.
\]

Since the coefficients of $(8/13)^{-1}$ in $\varphi_{mn}(r)$ decrease as $r$ increases, the remaining sum of $\varphi_{mn}(r)$ is $< 2.6\varphi_{mn}(4)$. Hence, (4.11) implies

\[
tr M < 0.8734 + 0.095(1.23077 + 0.05365 + 3.6(0.00728))
\]
\[< 0.8734 + 0.095(1.3107) < 0.998 < 1.
\]

This proves Theorem 4.1. We check directly for $m = 2, 3, 4$ that

\[
tr M_2(6) = 97/105, \quad tr M_3(14) = 0.95 + 1/7280,
\]
\[
tr M_4(25) = 0.87755 + 0.09359 + 0.0073 + 0.0000005 < 0.9719.
\]

This proves the parabolic bound $n > f(m) = (13m^2 - 5)/8$ to be sufficient for $tr M < 1$. Although some bound between this and $n > g(m)$ in (4.2) might also suffice for all $n$, the tight inequality (4.15) indicates that it would be difficult to prove.

5. The Bernoulli Continued Fraction. The entries $s_{i+j-2}(n)/n$ of the matrix $W_n(n)/n$ have as constant terms the Bernoulli numbers $B_{i+j}$ given in (1.2). The limit as $n \to 0$ of the leading principal minor of $W_n(n)/n$ is the determinant $b^*_m$ of order $m - 1$ expressible as

\[
b^*_m = \det(B_{i+j}) = \lim_{n \to 0} (nb_{11})(n^{-m}w_m(n)).
\]

Recalling $b_{11}$ from (4.3), $w_m(n)$ from (1.3), $v_m$ from (2.3) and $h_m$ from (2.4), we have

\[
\lim_{n \to 0} nb_{11} = {m \choose m}m^{-1}(m-1)^{m-1} = (-1)^{m-1}m,
\]
\[
\lim_{n \to 0} n^{-m}w_m(n) = h_m v_m^2 (-1)^m(m^{-1})^{m-1/2},
\]
\begin{align}
\text{(5.4)} \quad b^*_m &= (-1)^{m-1}(m-2)!m!u_m^2/m_2m, \\
\text{(5.5)} \quad b^*_m/b^*_{m-1} &= (-1)^{m-1}(m-1)!(m!)^4(m+1)/(2m)!(2m+1)!.
\end{align}

Since \( B_{i+j} = 0 \) for odd \( i + j \), we can rearrange rows and columns of the matrix \( (B_{i+j}) \) so the odd numbered ones precede the even numbered ones, and thus factor \( b^*_m \) as the product \( d_{m-1}d_{m-2} \) of two determinants, where

\[
\begin{vmatrix}
B_2 & B_4 & \cdots & B_{2k} \\
B_4 & B_6 & \cdots & B_{2k+2} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k} & B_{2k+2} & \cdots & B_{4k-2}
\end{vmatrix}
\]

\[
\begin{vmatrix}
B_4 & B_6 & \cdots & B_{2k+2} \\
B_6 & B_8 & \cdots & B_{2k+4} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k+2} & B_{2k+4} & \cdots & B_{4k}
\end{vmatrix}
\]

\[
\text{(5.6)}
\]

\[
\begin{vmatrix}
B_2 & B_4 & \cdots & B_{2k} \\
B_4 & B_6 & \cdots & B_{2k+2} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k} & B_{2k+2} & \cdots & B_{4k-2}
\end{vmatrix}
\]

\[
\begin{vmatrix}
B_4 & B_6 & \cdots & B_{2k+2} \\
B_6 & B_8 & \cdots & B_{2k+4} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k+2} & B_{2k+4} & \cdots & B_{4k}
\end{vmatrix}
\]

\[
\text{(5.7)}
\]

\[
-d_{m-3}d_{m}/d_{m-1}d_{m-2} = (m-1)m^4(m+1)/(2m-1)(2m)2(2m+1)
\]

\[
= (1/4)((m-1)m/(2m-1))(m(m+1)/(2m+1)).
\]

**Theorem 5.1.** The divergent asymptotic alternating series

\[
\text{(5.9)} \quad B(x) = \sum_{k=1}^\infty B_{2k}(2x)^{2k} = 4x^2/6 - 16x^4/30 + 64x^6/42 \cdots
\]

has the convergent continued fraction expansion (1.6).

**Proof.** By the general theory of continued fractions [2], [9], if a formal power series (5.9) with arbitrary coefficients \( B_{2k} \) is expanded into continued fractions of the form

\[
\begin{array}{c}
\frac{a_1(2x)^2}{1 + } \quad \frac{a_2(2x)^2}{1 + } \\
\frac{x^2/c_0}{c_1 + } \quad \frac{x^2}{c_2 + } \quad \frac{x^2}{c_3 + } \quad \cdots
\end{array}
\]

and if the \( d_k \)'s are defined by (5.6), then
(5.11) \[ a_m = \frac{1}{4c_{m-1}c_m} = -\frac{d_{m-2}d_m}{d_{m-2}d_{m-1}}, \quad m \geq 1. \]

For the Bernoulli series Eqs. (5.5) and (5.11) imply

(5.12) \[ c_m = \frac{m(m + 1)}{(2m + 1)} = \frac{1}{m} + \frac{1}{m + 1}, \quad m \geq 1, \]

while the condition \( 1/c_0c_1 = 4B_1 = 2/3 \) implies \( c_0 = 1. \) Since \( \Sigma c_m \) is divergent, the continued fraction (1.6) converges, and Theorem 5.1 is proved.

We can apply this continued fraction to approximate \( \pi^2. \) It would require about a billion terms of the series \( \Sigma_1^\infty (1/k^2) \) to approximate \( \pi^2/6 \) to nine decimals. But the Euler-Maclaurin summation formula gives the remainder after 5 terms by the expression

(5.13) \[ \int_1^\infty x^{-2} \, dx + 1/2 \cdot 6^2 + \sum_{k=1}^\infty B_{2k}(1/6)^{2k+1}. \]

This alternating series diverges, with minimum remainder of about \( 10^{-15} \) after the 19th term. Using the convergent continued fraction instead, we have

(5.14) \[ \pi^2 = 6(1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6 + 1/72) + B(1/12) \]

\[ = 9.865 + \left| \frac{12^{-2}}{1 + 1/2} \right| + \left| \frac{12^{-2}}{1/2 + 1/3} \right| + \left| \frac{12^{-2}}{1/3 + 1/4} \right|, \]

(5.15) \[ \pi^2 = 9.865 + \frac{1}{12 + 6} + \frac{1}{6 + 4} + \frac{1}{4 + 3} + \frac{1}{3 + 2.4} + \frac{1}{2.4 + 2} + \frac{1}{2 + r}, \]

where the sixth convergent with \( r = 12/7 \) has an error about \( 10^{-12}, \) and the tenth convergent (which changes this \( r \) to 1.9976) has an error less than \( 10^{-15}, \) giving \( \pi^2 = 9.869604401089359. \)

The function \( s^{-1}B(s^{-1}) \) is the Laplace transform of \( x \coth x - 1. \)

Continued fractions for the Laplace transforms of \( \tanh x, \text{sech} x \) and \( x \text{csch} x \) can also be obtained by similar methods, but have already been derived by Stieltjes [8] and others, and are listed by Wall [9, p. 369]. The author has not found the continued fraction (1.6) in the literature, nor the determinantal formula (5.4) which evaluates the first principal \( m \times m \) minor \( b^*_m = |B_{i+j}|, i, j = 1, \ldots, (\text{omitting} B_0 \text{and} B_1) \) of the determinants \( |B_{i+j-2}| \) of order \( m + 1 \) called \( \Delta_m(B) \) by Al-Salam and Carlitz [1, p. 93, (3.1)] which in the notation of (2.3) becomes

(5.16) \[ \Delta_m(B) = (-1)^{m(m+1)/2}(m!!)^6/(2m + 1)!!. \]

Comparing (5.16) with (5.4) for order \( m, \) we have

(5.17) \[ |B_{i+j}|_m = (-1)^m(m + 1)|B_{i+j-2}|_{m+1}. \]