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The Hankel Power Sum Matrix Inverse and the Bernoulli Continued Fraction

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Abstract. The $m \times m$ Hankel power sum matrix $W = VV^T$ (where V is the $m \times n$ Vandermonde matrix) has (i, j)-entry $S_{i+j-2}(n)$, where $S_p(n) = \Sigma_{k=1}^n k^p$. In solving a statistical problem on curve fitting it was required to determine $f(m)$ so that for $n > f(m)$ all eigenvalues of W^{-1} would be less than 1. It is proved, after calculating W^{-1} by first factoring W into easily invertible factors, that $f(m) = (13m^2 - 5)/8$ suffices. As by-products of the proof, close approximations are given for the Hilbert determinant, and a convergent continued fraction with mth partial denominator $m^{-1} + (m+1)^{-1}$ is found for the divergent Bernoulli number series $\Sigma B_{2k}(2x)^{2k}$.

1. Introduction. Defined as the product $W = VV^T$ of the $m \times n$ Vandermonde matrix $V = (j^{i-1})$ with its transpose V^T, the $m \times m$ Hankel power sum matrix $W = W_m$ has (i, j)-entry $S_{i+j-2}(n)$, where

$$(1.1) \qquad S_p = S_p(n) = \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{[p/2]} \frac{B_{2k}}{2k} \binom{p}{2k-1},$$

and where B_{2k} are the Bernoulli numbers [3], [6]:

$$(1.2) \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots .$$

In solving a statistical problem involving the fitting of polynomial curves of degree m to $n > m$ points, for increasing m and n, it was required [4] to find a function $f(m)$, such that, whenever $n > f(m)$, the eigenvalues μ_k of $M = W^{-1}$ would all be less than 1. We evaluate $w_m = \det W_m$ in Section 2 as

$$(1.3) \qquad w_m = \det W_m = h_m \prod_{i,j=1}^{m} (n + i - j),$$

where h_m is the determinant of the Hilbert matrix H_m, and we obtain close estimates for h_m. In Section 3 we factor W_m as a product of easily invertible matrices of which only diagonal matrices involve n, and we also explicitly invert W_m and H_m. In Section 4 we estimate the trace of $M = W^{-1}$ and find that the function

$$(1.4) \qquad f(m) = (13m^2 - 5)/8$$

suffices for powers of M to converge when $n > f(m)$.

As a by-product of this investigation, we find in Section 5 that the divergent

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asymptotic series $B(x)$ with Bernoulli number coefficients

$$B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k}, \tag{1.5}$$

related to the Laplace transform of $x \coth x - 1$, has the convergent continued fraction expansion

$$B(x) = \frac{x^2|}{|1 + 1/2 +} \; \frac{x^2|}{|1/2 + 1/3 +} \; \frac{x^2|}{|1/3 + 1/4 +} \cdots . \tag{1.6}$$

In fact, $B(1/12) = \pi^2 - 9.865$ is given with error $< 2 \times 10^{-12}$ by the sixth convergent of this continued fraction.

2. The Determinants. Since $S_0(n) = n$ and $n(n + 1)/2$ divides $S_k(n)$ for $k > 0$, it follows directly that $w_m = \det W_m$ has the algebraic factor $n^m(n + 1)^{m-1}$. For $r < m$, $(n - r)^{m-r}$ is also an algebraic factor of w_m, since the matrix $W_m(n)$ has rank r and nullity $m - r$, when $n = r$ for $r = 1, 2, \ldots, m - 1$. Since the polynomials $S_p(n)$ are generated by the function

$$G(x, n) = (e^{xn} - 1)/(1 - e^{-x}) = \sum_{p=0}^{\infty} S_p(n)x^p/p!, \tag{2.1}$$

we see from the identity

$$G(x, n) + G(-x, -n - 1) + 1 = 0$$

that

$$S_p(n) + (-1)^p S_p(-n - 1) + \delta_{p,0} = 0. \tag{2.2}$$

Hence, w_m has the algebraic factor $(n + 1 + r)^{m-1-r}$. We have found m^2 linear functions of n as factors of the polynomial $w_m(n)$ which is of degree m^2 in n. The remaining factor is the determinant of the leading coefficients $1/(i + j - 1)$, namely the determinant h_m of the ill-conditioned Hilbert matrix H_m of order m [5], [7]. This proves Eq. (1.3).

If we take $m = n$ in (1.3), the Vandermonde matrix V in $W = VV^T$ is square. Its determinant v_m is

$$v_m = \det V_m = 1!\ 2!\ 3! \cdots (m - 1)! \equiv (m - 1)!!. \tag{2.3}$$

Hence by (1.3) and (2.3)

$$h_m = \det V_m^2 \Big/ \prod_{i,j=1}^{m} (m + i - j) = v_m^4/v_{2m}. \tag{2.4}$$

The ratio of successive h_m's is

$$h_m/h_{m+1} = (2m + 1)!(2m)!/(m!)^4 = (2m + 1)\binom{2m}{m}^2. \tag{2.5}$$

The following theorem gives a close approximation for h_m.

THEOREM 2.1. *The determinant h_m of the Hilbert matrix H_m of order m is* *given to* 10 *significant figures for* $m > 4$ *by*

$$h_m = 4^{-m(m-1)}(\pi/2)^{m-1}m^{-1/4}\exp R_m, \tag{2.6}$$

where the remainder function R_m *is defined by*

$$R_m = \int_0^\infty (e^{-2t} - e^{-2mt}) \tanh^2 (t/2)(4t)^{-1}\, dt \tag{2.7}$$

and is approximated to 9 *decimals for* $m \geqslant 5$ *by*

$$\begin{aligned} R_m &= 0.013081539 - 2^{-6}m^{-2} + 2^{-8}m^{-4} - 2^{-8.5}m^{-6} \\ &\quad + 2^{-8}m^{-8} - 2^{-7}m^{-10}. \end{aligned} \tag{2.8}$$

Proof. The ratio h_m/h_{m+1} in (2.5) is related to the Wallis approximation π_m for π by

$$\frac{\pi_m}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\cdots\frac{2m}{2m-1}\,\frac{2m}{2m+1} = 2^{4m}h_{m+1}/h_m. \tag{2.9}$$

If we express $\ln n$ in the form

$$\ln n = \int_1^n \frac{ds}{s} = \int_0^\infty \int_1^n e^{-st}\, ds\, dt = \int_0^\infty (e^{-t} - e^{-nt})t^{-1}\, dt, \tag{2.10}$$

then $\ln(\pi_m/2)$ and its limit $\ln(\pi/2)$ are expressible as

$$\begin{aligned} \ln(\pi_m/2) &= \int_0^\infty (e^{-t} - 2e^{-2t} + 2e^{-3t} - 2e^{-2mt} + e^{-(2m+1)t})t^{-1}\, dt \\ &= \int_0^\infty (e^{-t} - e^{-(2m+1)t})(1 - e^{-t})(1 + e^{-t})^{-1}t^{-1}\, dt, \end{aligned} \tag{2.11}$$

$$\ln(\pi/2) = \int_0^\infty e^{-t} \tanh (t/2)t^{-1}\, dt, \tag{2.12}$$

$$\ln(\pi/\pi_k) = \int_0^\infty e^{-(2k+1)t} \tanh (t/2)t^{-1}\, dt, \tag{2.13}$$

$$(\pi/2)^{m-1} \prod_{k=1}^{m-1} (\pi_k/\pi) = 2^{2m(m-1)}h_m/h_1. \tag{2.14}$$

Summing in (2.13) from $k = 1$ to $m - 1$ yields

$$\begin{aligned} \ln[(\pi/2)^{m-1}2^{-2m(m-1)}/h_m] &= \int_0^\infty (e^{-2t} - e^{-2mt})(e^{t/2} + e^{-t/2})^{-2}t^{-1}\, dt \\ &= (1/4)\ln(2m/2) - R_m \end{aligned} \tag{2.15}$$

by (2.10), where R_m is defined by (2.7). Equation (2.6) follows from (2.15). To obtain (2.8) we evaluate $R_4 = .012119610988$ from (2.6) setting $h_4 = 1/6048000$ in (2.6). Then we compute $R_\infty - R_m$ from (2.7) by replacing $\tanh(t/2)$ by the first five terms of its series, and set $m = 4$ to get R_∞ in (2.8). We check the tenth decimal by working from h_5 instead. This gives $\exp R_m$ and h_m accurate to 10 significant figures.

For $m = 20$ we find $R_{20} = .0130425009$ and

$$h_{20} = 4.206178954 \times 10^{-226}. \tag{2.16}$$

The matrices H_m and $W_m(n)$ are ill conditioned. In fact, $W_3(3)$ has the eigenvalues $\lambda_1 = 113.4132$, $\lambda_2 = 1.564253$, $\lambda_3 = .02254695$ and the conditioning ratio $\lambda_1/\lambda_3 = 5030$. So the usual computer methods for inverting $W_m(n)$ are unreliable [5], [7].

3. Inversion by Factoring. To invert the ill-conditioned $m \times m$ matrix $W = W_m(n)$ with (i, j)-entry $S_{i+j-2}(n)$, we first factor it into easily invertible factors, restricting the variable n to diagonal matrix factors $EP = (\operatorname{diag} e_i p_i)$ and $Q = (\operatorname{diag} q_j)$, where

$$e_i = (-1)^{i-1}, \quad p_i = \binom{n+i-1}{n-m}, \quad q_j = \binom{n-j}{n-m}. \tag{3.1}$$

We denote by $T = (t_{ij})$ the lower Pascal triangle matrix with

$$t_{ij} = \binom{i-1}{j-1} = \binom{i-1}{i-j} = (-1)^{i+j}\binom{-j}{i-j}. \tag{3.2}$$

We note that ETE has entries $\binom{-j}{i-j}$, so

$$(TETE)_{ij} = \sum_{k=j}^{i} \binom{i-1}{i-k}\binom{-j}{k-j} = \binom{i-j-1}{i-j} = \delta_{ij} \tag{3.3}$$

and $T^{-1} = ETE$. Next, we define an $m \times m$ lower triangular row stochastic matrix $A = (a_{ij})$ that converts the integral powers in V into binomial coefficients by the formula

$$(AV)_{ik} = \sum_{r=1}^{i} a_{ir} k^{r-1} = \binom{k+i-2}{i-1}, \qquad k = 1, 2, \ldots, n. \tag{3.4}$$

The a_{ir} are related to Stirling numbers of the first kind, and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2! & 1/2! & 0 & 0 & \cdots \\ 0 & 2/3! & 3/3! & 1/3! & 0 & \cdots \\ 0 & 6/4! & 11/4! & 6/4! & 1/4! & \cdots \end{bmatrix} = (a_{ij}). \tag{3.5}$$

From (3.5) and (3.4) we obtain

$$(AV)_{ik} = \binom{i+k-2}{i-1} = \sum_{j=1}^{i} \binom{i-1}{i-j}\binom{k-1}{j-1} = \sum_{j=1}^{k} t_{ij}\binom{k-1}{j-1}. \tag{3.6}$$

To factor W we now write

$$(AV(T^{-1}AV)^T)_{ij} = \sum_{k=j}^{n} \binom{k+i-2}{i-1}\binom{k-1}{j-1} \tag{3.7}$$
$$= \binom{i+j-2}{i-1} \sum_{k=j}^{n} \binom{k+i-2}{j+i-2}.$$

Summing over k yields

$$(AW(T^{-1}A)^T)_{ij} = \binom{i+j-2}{i-1}\binom{n+i-1}{j+i-1} \tag{3.8}$$
$$= p_i\binom{i+j-2}{i-1}\binom{m+i-1}{j+i-1}\Big/ q_j.$$

THEOREM 3.1. *The inverse matrix $M = W^{-1}$ has the factorization*

$$M = W^{-1} = A^T EBEA, \quad B = T^T QTT^T TP^{-1}, \tag{3.9}$$

where E, P, Q, T, A are defined in (3.1), (3.2) *and* (3.5).

Proof. Entries of TT^T and $TT^T T$ are

$$(TT^T)_{ik} = \sum_{r=1}^{i} \binom{i-1}{i-r}\binom{k-1}{r-1} = \binom{i+k-2}{i-1}, \tag{3.10}$$

$$(TT^T T)_{ij} = \sum_{k=j}^{m} \binom{i+k-2}{k-1}\binom{k-1}{j-1} = \binom{i+j-2}{i-1} \sum_{k=j}^{m} \binom{k+i-2}{j+i-2}. \tag{3.11}$$

Summing over k yields

$$(TT^T T)_{ij} = \binom{i+j-2}{i-1}\binom{m+i-1}{j+i-1}. \tag{3.12}$$

Combining (3.8) and (3.12), we have

$$AWA^T = PTT^T TQ^{-1} T^T. \tag{3.13}$$

Since the diagonal sign matrix E commutes with P and Q but transforms T and T^T into their inverses,

$$(AWA^T)^{-1} = ET^T QTT^T TP^{-1} E = EBE \tag{3.14}$$

and (3.9) is proved.

Equation (3.12) provides a simple method for inverting the Hilbert matrix H.

THEOREM 3.2. *The inverse of the $m \times m$ Hilbert matrix $H = (h_{ij})$ with (i, j)-entry $h_{ij} = 1/(i + j - 1)$ is given by*

$$(H^{-1})_{ij} = d_i d_i' h_{ij} d_j d_j', \tag{3.15a}$$

$$d_i' = \binom{-m-1}{i-1}, \quad d_j = m\binom{m-1}{j-1}. \tag{3.15b}$$

Proof. Factoring (3.12) yields

$$(3.16)\qquad (ETT^TT)_{ij} = (-1)^{i-1}\binom{m+i-1}{i-1}\frac{m}{i+j-1}\binom{m-1}{j-1} = d_i' h_{ij} d_j.$$

Since ETT^TT is involutory, $D'HD = (D'HD)^{-1}$. Also,

$$(3.17)\qquad h_m = \det H_m = \pm 1\Big/\prod_{i=1}^{m} d_i d_i'.$$

Although the matrix B in (3.9) is symmetric, its symmetry is not obvious from formula (3.9).

THEOREM 3.3. *The symmetric matrix $B = (b_{ij})$ in* (3.9) *has entries expressible in terms of descending factorials $(x)_r = x(x-1)\cdots(x-r+1)$ as follows:*

$$(3.18)\qquad b_{ij} = \sum_{r\geqslant i+j-1} \frac{(m+i-1)_r(m+j-1)_r}{(n-m+r)_r r!}\binom{r-1}{i-1,\,j-1},$$

where $\binom{r-1}{i-1,j-1}$ is the trinomial coefficient

$$\binom{r-1}{i+j-2}\binom{i+j-2}{i-1}.$$

Proof. To transform B in (3.9) we evaluate

$$(T^TQTT^T)_{is} = \sum_{r=i}^{m}\binom{r-1}{i-1}\binom{n-r}{n-m}\binom{r+s-2}{r-1}$$

$$(3.19)\qquad = \binom{i+s-2}{i-1}\sum_{r=i}^{m}\binom{r+s-2}{i+s-2}\binom{n-r}{n-m}$$

$$= \binom{i+s-2}{i-1}\binom{n+s-1}{m-i},$$

$$(BP)_{ij} = \sum_{s=j}^{m}\binom{i+s-2}{i-1}\binom{n+s-1}{m-i}\binom{s-1}{j-1}$$

$$(3.20)\qquad = \binom{i+j-2}{i-1}\sum_{s=j}^{m}\binom{i+s-2}{s-j}\sum_{r\geqslant i+j-1}\binom{s-j}{r-i-j+1}\binom{n+j-1}{m+j-1-r}$$

$$= \sum_{r\geqslant i+j-1}\sum_{s=j}^{m}\binom{i+s-2}{r-1}\binom{r-1}{i-1,\,j-1}\binom{n+j-1}{n-m+r}.$$

Summing over s and dividing by p_j, we have

$$(3.21)\qquad b_{ij} = \sum_{r\geqslant i+j-1}\binom{m+i-1}{r}\binom{r-1}{i-1,\,j-1}\binom{m+j-1}{r}\Big/\binom{n-m+r}{r}.$$

Writing $(x)_r = r!\binom{x}{r}$, Eq. (3.21) becomes (3.18).

To conserve space in displaying the symmetric matrices $M_m(n) = W_m^{-1}(n)$ we show the upper half of M_3 and the lower half of M_4.

(3.22)

$$\begin{array}{c} M_4 \qquad\qquad\qquad\qquad\qquad\qquad M_3 \end{array}$$

$$\begin{bmatrix} \frac{16n^3 + 24n^2 + 56n + 24}{(n)_4} & \frac{9(n^2 + n) + 6}{(n)_3} & \frac{-18(2n + 1)}{(n)_3} & \frac{30}{(n)_3} \\ \frac{-120(n^2 + n) - 100}{(n)_4} & \frac{1200(n^4 + 4.5n^3 + 7n^2 + 5n + 11/6)}{(n + 3)_7} & \frac{(24n + 12)(8n + 1)}{(n + 2)_5} & \frac{-180(n + 1)}{(n + 2)_5} \\ \frac{120(2n + 1)}{(n)_4} & \frac{-300(n + 1)(3n + 2)(3n + 5)}{(n + 3)_7} & \frac{360(2n + 1)(9n + 13)}{(n + 2)_5} & \frac{180}{(n + 2)_5} \\ \frac{-140}{(n)_4} & \frac{280(6n^2 + 15n + 11)}{(n + 3)_7} & \frac{-4200(n + 1)}{(n + 3)_7} & \frac{2800}{(n + 3)_7} \end{bmatrix}.$$

4. Estimation of $\operatorname{tr} M$. Since W and M are positive definite for $n > m$, all eigenvalues μ_k of M will satisfy $\mu_k < 1$ if $\operatorname{tr} M \leqslant 1$, $m > 1$. For A in (3.5) and $e_i = (-1)^{i-1}$, the first and last diagonal entries of $M = M_m$ are b_{11} and $b_{mm}/((m-1)!)^2$. Numerical computation shows that the maximum n for which $\det(W_m(n) - I) = 0$ are given for $m = 1, 2, 3, 4$ by

$$(m, n) = (1, 1), (2, 5.82090), (3, 13.3776), (4, 24.24453). \tag{4.1}$$

The parabola through the first three points is

$$n = g(m) = 1.5679m^2 - .0828m - .4851, \tag{4.2}$$

and we find $g(4) = 24.270 > 24.24453$. A slightly higher value than (4.2) will be required for $\operatorname{tr} M \leqslant 1$. We first estimate the dominant diagonal entry b_{11} of M.

$$\begin{aligned} \binom{n}{m} b_{11} &= \sum_{r=1}^{m} \binom{m}{r}^2 \binom{n}{m} \bigg/ \binom{n-m+r}{r} = \sum_{r=1}^{m} \binom{m}{r} \binom{n}{m-r} \\ &= \binom{n+m}{m} - \binom{n}{m}. \end{aligned} \tag{4.3}$$

$$1 + b_{11} = \binom{n+m}{m} \bigg/ \binom{n}{m} = \prod_{k=1}^{m} \frac{2n + 1 + (2k-1)}{2n + 1 - (2k-1)}, \tag{4.4}$$

$$\ln(1 + b_{11}) = \sum_{k=1}^{m} \ln \frac{1 + (2k-1)/(2n+1)}{1 - (2k-1)/(2n+1)} = \sum_{r=1}^{\infty} \frac{\theta(m, r)}{(2r-1)(2n+1)^{2r-1}}, \tag{4.5a}$$

where

$$\theta(m, r) = \sum_{k=1}^{m} 2(2k-1)^{2r-1} < \int_0^{2m} x^{2r-1}\, dx = (2m)^{2r}/2r. \tag{4.5b}$$

We now assume the inequalities $n > f(m)$ in (1.4).

THEOREM 4.1. *The matrix* $M = W_m^{-1}(n)$ *has trace* < 1 *if*

$$n > 1.625m^2 - .625 \quad \text{and} \quad m \geqslant 5. \tag{4.6}$$

Proof. If (4.6) is satisfied for $m = 5$, then $n \geqslant 40$, and

(4.7a) $$b_{11} \leqslant (45)_5/(40)_5 - 1 = 62639/73112 = .856754.$$

If (4.6) is satisfied for $m \geqslant 6$, then $(2n+1)/2m^2 > 1.6215$ and

(4.8)
$$\begin{aligned}\ln(1+b_{11}) &< \frac{2m^2}{2n+1}\sum_{r=1}^{\infty}\left(\frac{2m}{2n+1}\right)^{2r-2}\Big/ r(2r-1)\\ &< \frac{1}{1.6215}\sum_{r=1}^{\infty}\left(\frac{1}{9.729}\right)^{2r-2}\Big/ r(2r-1) < .62777,\end{aligned}$$

(4.7b) $$b_{11} < .8734 \quad \text{for } m \geqslant 6,\, n > (13m^2-5)/8.$$

The rest of $\operatorname{tr} M$ is given by

(4.9) $$\operatorname{tr} M - b_{11} = \sum_{k=2}^{m}\ \sum_{i,j=k}^{m} a_{ik}(-1)^i b_{ij}(-1)^j a_{jk}.$$

We replace i, j, r by $i+1, j+1, r+2$ and write

(4.10) $$b_{i+1,j+1} = \sum_{k=1}^{m-1} y_{ij}^{(r)},\quad y_{ij}^{(r)} = \frac{(m+i)_{r+2}(m+j)_{r+2}}{(n-m+r+2)_{r+2}(r+2)!}\binom{r+1}{i,\,j} = y_{ji}^{(r)}.$$

Then

(4.11) $$\operatorname{tr} M - b_{11} = y_{11}^{(1)}\sum_{r=1}^{2m-1}\varphi_{mn}(r),\quad \varphi_{mn}(r) = \sum_{i+j=2}^{r+1} c_{ij}y_{ij}^{(r)}/y_{11}^{(1)},$$

where the entries of the $(m-1)\times(m-1)$ matrix $C = (c_{ij})$ are

(4.12a) $$c_{ij} = (-1)^{i+j}\sum_{k=1}^{m-1} a_{i+1,k}a_{j+1,k} = c_{ji},$$

(4.12b) $$C = \frac{1}{720}\begin{bmatrix} 720 & -360 & 240 & -180 & 144 & \cdots\\ -360 & 360 & -300 & 255 & -222 & \cdots\\ 240 & -300 & 280 & -255 & 233 & \cdots\\ -180 & 255 & -255 & 242.5 & -228.5 & \cdots\\ 144 & -222 & 233 & -228.5 & 220.1 & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

The dominant term $y_{11}^{(1)}$ satisfies

(4.13) $$y_{11}^{(1)} < \frac{(m+1)_3(m+1)_3/3}{(13m^2/8 - m + 19/8)_3} \leqslant \frac{6_3 6_3/3}{(38)_3} = \frac{200}{2109} < .094832,$$

since the rational function decreases for $5 \leqslant m$. The function $\varphi_{mn}(1)$ is 1, but for $r > 1$, then $\varphi_{mn}(r)$ in (4.11) are bounded by rational functions which increase for

$m \geqslant 5$, and which we replace by their limits as $m \longrightarrow \infty$.

$$\varphi_{mn}(2) = (y_{11}^{(2)} - y_{12}^{(2)})/y_{11}^{(1)} = 3(m-2)(m-6)/(13m^2 - 8m + 27)$$

(4.14a)

$$< 3/13 = .23077,$$

$$\varphi_{mn}(3) = (y_{11}^{(3)} - y_{12}^{(3)} + 2y_{13}^{(3)}/3 + y_{22}^{(3)}/2)/y_{11}^{(1)}$$

(4.14b) $$< 17(m^2 - 6m + 32)(m-2)(m-2.4)/(120)(13m^2/8 - m + 35/8)_2$$

$$< (17/120)(8/13)^2 = .05365.$$

Similar calculations yield

(4.14c) $$\varphi_{mn}(4) < (1/32)(8/13)^3 = .00728.$$

Since the coefficients of $(8/13)^{r-1}$ in $\varphi_{mn}(r)$ decrease as r increases, the remaining sum of $\varphi_{mn}(r)$ is $< 2.6\varphi_{mn}(4)$. Hence, (4.11) implies

(4.15)
$$\operatorname{tr} M < .8734 + .095(1.23077 + .05365 + 3.6(.00728))$$
$$< .8734 + .095(1.3107) < .998 < 1.$$

This proves Theorem 4.1. We check directly for $m = 2, 3, 4$ that

(4.16)
$$\operatorname{tr} M_2(6) = 97/105, \quad \operatorname{tr} M_3(14) = .95 + 1/7280,$$
$$\operatorname{tr} M_4(25) = .87755 + .09359 + .0073 + .0000005 < .9719.$$

This proves the parabolic bound $n > f(m) = (13m^2 - 5)/8$ to be sufficient for $\operatorname{tr} M < 1$. Although some bound between this and $n > g(m)$ in (4.2) might also suffice for all n, the tight inequality (4.15) indicates that it would be difficult to prove.

5. The Bernoulli Continued Fraction. The entries $S_{i+j-2}(n)/n$ of the matrix $W_m(n)/n$ have as constant terms the Bernoulli numbers B_{i+j-2} given in (1.2). The limit as $n \longrightarrow 0$ of the leading principal minor of $W_m(n)/n$ is the determinant b_{m-1}^* of order $m-1$ expressible as

(5.1) $$b_{m-1}^* = \det(B_{i+j}) = \lim_{n \to 0} (nb_{11})(n^{-m}w_m(n)).$$

Recalling b_{11} from (4.3), $w_m(n)$ from (1.3), v_m from (2.3) and h_m from (2.4), we have

(5.2) $$\lim_{n=0} nb_{11} = \binom{m}{m} m \Big/ \binom{-1}{m-1} = (-1)^{m-1} m,$$

(5.3) $$\lim_{n=0} n^{-m} w_m(n) = h_m v_m^2 (-1)^{m(m-1)/2},$$

$$b^*_{m-1} = (-1)^{(m-1)(m-2)/2} m v_m^6 / v_{2m}, \tag{5.4}$$

$$b^*_m / b^*_{m-1} = (-1)^{m-1}(m-1)!(m!)^4(m+1)!/(2m)!(2m+1)!. \tag{5.5}$$

Since $B_{i+j} = 0$ for odd $i + j$, we can rearrange rows and columns of the matrix (B_{i+j}) so the odd numbered ones precede the even numbered ones, and thus factor b^*_{m-1} as the product $d_{m-1}d_{m-2}$ of two determinants, where

$$d_{2k-1} = \begin{vmatrix} B_2 & B_4 & \cdots & B_{2k} \\ B_4 & B_6 & \cdots & B_{2k+2} \\ \cdot & \cdot & \cdot & \cdot \\ B_{2k} & B_{2k+2} & \cdots & B_{4k-2} \end{vmatrix}, \tag{5.6}$$

$$d_{2k} = \begin{vmatrix} B_4 & B_6 & \cdots & B_{2k+2} \\ B_6 & B_8 & \cdots & B_{2k+4} \\ \cdot & \cdot & \cdot & \cdot \\ B_{2k+2} & B_{2k+4} & \cdots & B_{4k} \end{vmatrix},$$

$$d_m / d_{m-2} = b^*_m / b^*_{m-1}, \tag{5.7}$$

$$\begin{aligned} -d_{m-3}d_m/d_{m-1}d_{m-2} &= (m-1)m^4(m+1)/(2m-1)(2m)^2(2m+1) \\ &= (1/4)((m-1)m/(2m-1))(m(m+1)/(2m+1)). \end{aligned} \tag{5.8}$$

THEOREM 5.1. *The divergent asymptotic alternating series*

$$B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k} = 4x^2/6 - 16x^4/30 + 64x^6/42 \cdots \tag{5.9}$$

has the convergent continued fraction expansion (1.6).

Proof. By the general theory of continued fractions [2], [9], if a formal power series (5.9) with arbitrary coefficients B_{2k} is expanded into continued fractions of the form

$$\frac{a_1(2x)^2|}{|\ 1+} \frac{a_2(2x)^2|}{|\ 1+} \cdots = \frac{x^2/c_0|}{|c_1 +} \frac{x^2\ |}{|c_2 +} \frac{x^2\ |}{|c_3 +} \cdots \tag{5.10}$$

and if the d_k's are defined by (5.6), then

(5.11) $$a_m = 1/4c_{m-1}c_m = -d_{m-3}d_m/d_{m-2}d_{m-1}, \quad m \geq 1.$$

For the Bernoulli series Eqs. (5.5) and (5.11) imply

(5.12) $$c_m = (m(m+1)/(2m+1))^{-1} = 1/m + 1/(m+1), \quad m \geq 1,$$

while the condition $1/c_0c_1 = 4B_1 = 2/3$ implies $c_0 = 1$. Since Σc_m is divergent, the continued fraction (1.6) converges, and Theorem 5.1 is proved.

We can apply this continued fraction to approximate π^2. It would require about a billion terms of the series $\Sigma_1^\infty(1/k^2)$ to approximate $\pi^2/6$ to nine decimals. But the Euler-Maclaurin summation formula gives the remainder after 5 terms by the expression

(5.13) $$\int_6^\infty x^{-2}\,dx + 1/2 \cdot 6^2 + \sum_{k=1}^{\infty} B_{2k}(1/6)^{2k+1}.$$

This alternating series diverges, with minimum remainder of about 10^{-15} after the 19th term. Using the convergent continued fraction instead, we have

(5.14) $$\begin{aligned}\pi^2 &= 6(1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6 + 1/72) + B(1/12)\\ &= 9.865 + \frac{12^{-2}|}{|1 + 1/2 +}\;\frac{12^{-2}|}{|1/2 + 1/3 +}\;\frac{12^{-2}|}{|1/3 + 1/4 +},\end{aligned}$$

(5.15) $$\begin{aligned}\pi^2 &= 9.865\\ &+ \frac{1/12|}{|12 + 6 +}\;\frac{1|}{|6 + 4 +}\;\frac{1|}{|4 + 3 +}\;\frac{1\ \ |\ |}{|3 + 2.4 +}\;\frac{1|}{|2.4 + 2 +}\;\frac{1|}{|2 + r},\end{aligned}$$

where the sixth convergent with $r = 12/7$ has an error about 10^{-12}, and the tenth convergent (which changes this r to 1.9976) has an error less than 10^{-15}, giving $\pi^2 = 9.869604401089359$.

The function $s^{-1}B(s^{-1})$ is the Laplace transform of $x \coth x - 1$.

Continued fractions for the Laplace transforms of $\tanh x$, $\operatorname{sech} x$ and $x \operatorname{csch} x$ can also be obtained by similar methods, but have already been derived by Stieltjes [8] and others, and are listed by Wall [9, p. 369]. The author has not found the continued fraction (1.6) in the literature, nor the determinantal formula (5.4) which evaluates the first principal $m \times m$ minor $b_m^* = |B_{i+j}|$, $i, j = 1, \ldots,$ (omitting B_0 and B_1) of the determinant $|B_{i+j-2}|$ of order $m + 1$ called $\Delta_m(B)$ by Al-Salam and Carlitz [1, p. 93, (3.1)] which in the notation of (2.3) becomes

(5.16) $$\Delta_m(B) = (-1)^{m(m+1)/2}(m!!)^6/(2m+1)!!.$$

Comparing (5.16) with (5.4) for order m, we have

(5.17) $$|B_{i+j}|_m = (-1)^m(m+1)|B_{i+j-2}|_{m+1}.$$

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1. W. A. AL-SALAM & L. CARLITZ, "Some determinants of Bernoulli, Euler, and related numbers," *Portugal. Math.*, v. 18, 1959, pp. 91–99.

2. J. S. FRAME, "The solution of equations by continued fractions," *Amer. Math. Monthly*, v. 60, 1953, pp. 293–305.

3. J. S. FRAME, "Bernoulli numbers modulo 27000," *Amer. Math. Monthly*, v. 68, 1961, pp. 87–95.

4. D. C. GILLILAND & JAMES HANNAN, *Detection of Singularities in the Countable General Linear Model*, Department of Statistics, Michigan State University, RM-217, DCG-8, JH-10, Aug. 1971.

5. E. ISAACSON & H. B. KELLER, *Analysis of Numerical Methods*, Wiley, New York, 1966, pp. 196, 217–218.

6. N. E. NÖRLUND, *Vorlesung über Differenzenrechnung*, Springer, Berlin, 1924, p. 18.

7. G. M. PHILLIPS & P. J. TAYLOR, *Theory and Application of Numerical Analysis*, Academic Press, New York, 1973, pp. 91, 246.

8. T. J. STIELTJES, "Sur quelques intégrales définies et leur dévéloppement en fractions continues," *Oeuvres Complètes*, vol. 2, P. Noordhoff, Groningen, 1918, pp. 378–391.

9. H. S. WALL, *Analytic Theory of Continued Fractions*, D. Van Nostrand, Princeton, N. J., 1948, pp. 369–376.