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Quadratic Fields With 3-Rank Equal to 4

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Abstract. In [2] there is reference to 119 known imaginary quadratic fields that have 3-rank $r \geqslant 4$. We examine these fields and determine the exact values of r. Their associated real fields and the distribution of their 3-Sylow subgroups are also studied. Some of the class groups are recorded since they are of special interest. These include examples having an infinite class field tower and only one ramified prime, and others having an infinite tower because of two different components of their class groups.

1. The Uncertainty Resolved. Following Scholz [1] we designate the 3-rank of a quadratic field as r if its discriminant d is negative and as s if its discriminant is positive. In [2] there are listed thirteen d from $d = -653329427$ to $d = -9906365947$, inclusive, for which $K = Q(\sqrt{d})$ has $r = 4$. For the two smallest d here the class group of K was also given. For 119 additional d it was indicated that K had $r \geqslant 4$ and that "probably" $r = 4$ held in all 119 cases. These d lie between $d = -10647173399$ and $d = -2908807157867$. "Probably" meant that there was evidence that made $r = 4$ the most probable value, but this evidence was not conclusive.

No $Q(\sqrt{d})$ is known for which $r > 4$. It is unknown if such a field exists, but we conjecture that it does. It is, therefore, very desirable to eliminate the uncertainty above and to determine whether any of these 119 K has $r \geqslant 5$. Let K' be the *associated real field* of K, that is, $K' = Q(\sqrt{D})$ where $D = -3d$ if $3 \nmid d$ and $D = -d/3$ if $3 \mid d$. By Scholz's theorem [1] the 3-ranks of K and K' satisfy

$$r = s \quad \text{or} \quad r = s + 1. \tag{1}$$

The condition on the right we call the *escalatory case.* If any of the 119 K has $r \geqslant 5$, its K' has $s \geqslant 4$.

Until very recently, no $Q(\sqrt{D})$ with $s > 3$ was known to exist, but now it is known [3] that infinitely many such D can be constructed. However, the smallest known of these, which we discuss briefly in Section 3, has $D > 10^{103}$. This is far too large if we wish to determine the class group, fundamental unit, etc. of $Q(\sqrt{D})$. We would like a smaller example, and if any of the 119 K above has $r \geqslant 5$, its K' with $s \geqslant 4$ would also be very welcome.

To minimize the computation needed to resolve the uncertainty we decided to proceed as follows:

I. With existing programs on an IBM-370-168, compute the class number h for each K' from

$$2hR/\sqrt{D} = L(1) \tag{2}$$

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by computing its regulator R and estimating its Dirichlet function $L(1)$ with sufficient accuracy from its Euler product.

II. If

$$81 \nmid h, \tag{3}$$

we must have $s = 3$ and $r = 4$. That occurred in 99 of the 119 cases.

III. If $81 \mid h$ (20 cases), we compute the class group of K with existing programs on a pocket calculator HP-67. We found that $r = 4$ occurred in all 20 cases.

IV. For these 20 K we determine that we are in the escalatory case by using known criteria of Scholz [1]. That was true, and so each K' has $s = 3$.

V. Further, the original 13 fields in [2] also have K' with $s = 3$.

So the uncertainty is resolved, and there still is no known example of $r > 4$. We comment on that briefly below.

2. The Best Laid Schemes; Quadratic Fields of Interest. The best laid schemes can be deflected [4] by unforeseen circumstances, and we did not strictly follow the economical plan with phases I through V indicated above. Some of the D nearly equal 10^{13} and the available programs for computing R and the Euler product in phase I had to be modified to handle D that are that large. On the other hand, composition of quadratic forms of discriminant d can be carried out [5] with numbers that are usually $< |d|^{1/2}$; and therefore, these K with 13 decimal d go nicely on a little HP-67 even though it only computes to 10 decimals. So before phase I could be completed most of the 119 class groups were computed, and phase III actually comprised only those of the 20 cases that had not yet been done. Eventually, all 119 + 13 class groups were computed on an HP-67 using the method described in [6]. Although the 132 K have class numbers that range between 5670 and 773550, their calculation is much accelerated [6, p. 417] by the knowledge that they all have $r \geqslant 4$.

Of the 132 K here, some are of sufficient mathematical significance that they should be recorded.

(A) Thirteen of the d are prime. We list $-d$ and the class group G of K in Table 1, where, as usual, $n \times m \times \cdots$ means a product of cyclic groups: $C(n) \times C(m) \times \cdots$.

TABLE 1

$-d$	G
4724490703	$3 \times 3 \times 3 \times 795$
13116019171	$3 \times 3 \times 3 \times 1035$
23095449499	$3 \times 3 \times 3 \times 1311$
115372694551	$3 \times 3 \times 3 \times 7623$
148484670259	$3 \times 3 \times 3 \times 2715$
226293460843	$3 \times 3 \times 3 \times 2085$
235145409907	$3 \times 3 \times 3 \times 1665$
282910884511	$3 \times 3 \times 3 \times 17037$
474077832979	$3 \times 3 \times 3 \times 4629$
597541961299	$3 \times 3 \times 3 \times 9627$
699234050083	$3 \times 3 \times 3 \times 4275$
936658298011	$3 \times 3 \times 3 \times 5289$
1571310110659	$3 \times 3 \times 3 \times 7095$

These 13 K have an infinite class field tower, cf. [7], while having *only one ramified prime.* Previously [7], one knew of $Q(\sqrt{-p})$ for the prime radicand $p =$ 83309629817 but since $p \equiv 1 \pmod 4$, this field has $d = -4p$ and 2 also ramifies.

Each of these 13 fields has $-d \equiv 4 \pmod 9$. The meaning of this is not really clear to us. The method used [2] tends to favor the residue class $-d \equiv 4 \pmod 9$. We do not ascribe any absolute significance to this striking uniformity and presume that prime d in other residue classes and with $r = 4$ will be discovered in due course.

(B) Two of the K have a 2-rank = 5 since they have six ramified primes. Both $r \geqslant 4$ and a 2-rank $\geqslant 5$ imply an infinite tower, so these two K have an infinite class field tower because of *two different components* of their class groups, cf. [7]. We record them in Table 2.

TABLE 2

$-d$	G
$8 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 1034639$	$2 \times 6 \times 6 \times 6 \times 150$
$8 \cdot 23 \cdot 31 \cdot 43 \cdot 131 \cdot 18131$	$2 \times 6 \times 6 \times 12 \times 168$

(C) Two pairs of K have isomorphic groups. They may perhaps be of use in certain investigations so we record them in Table 3.

TABLE 3

$-d$	G
$8 \cdot 863248159$	$3 \times 3 \times 3 \times 1320$
$7151 \cdot 8498573$	$3 \times 3 \times 3 \times 1320$
$4 \cdot 173 \cdot 47976209$	$3 \times 3 \times 6 \times 564$
$8 \cdot 14387 \cdot 358297$	$3 \times 3 \times 6 \times 564$

(D) Here is how the 132 3-Sylow subgroups are distributed, cf. [8].

$3 \times 3 \times 3 \times 3$	80 cases,	$3 \times 3 \times 3 \times 81$	5 cases,
$3 \times 3 \times 3 \times 9$	29 cases,	$3 \times 3 \times 3 \times 243$	2 cases,
$3 \times 3 \times 3 \times 27$	15 cases,	$3 \times 3 \times 9 \times 9$	1 case.

Since the last subgroup is so rare we should record this field:

$-d$	G
$41 \cdot 1827827279$	$3 \times 3 \times 9 \times 1854$

Since this G has $3^6 | h$, while no generator of its 3-Sylow subgroup has order $> 3^2$, it looked for an exciting 5 minutes as if it were the long-sought example of $r = 5$.

3. **Commentary.** With this last point we return to the original question. We were not surprised that all 119 K had $r = 4$ since there already was evidence that that was the case. Further, there is a heuristic argument [8] that the probability of $r = 5$ greatly decreases if

$$|d| < L_5 = \left(\frac{3^5 - 1}{2}\right)^6 \cdot \frac{27}{16} = 5.296 \times 10^{12},$$

and all 132 K here satisfy that inequality. There may well be examples of $r = 5$ below L_5 but they are hard to find since $|d|$ is so large. Of the 132 examples of $r = 4$

here, only seven satisfy

$$|d| < L_4 = \left(\frac{3^4 - 1}{2}\right)^6 \frac{27}{16} = 6.912 \times 10^9.$$

We should stress the fact that the 132 K here *do not* constitute all cases of $r = 4$ that occur up to $|d| < 2.909 \times 10^{12}$. Two older examples [7] fall in that range as do two recent examples by Solderitsch [9]. No doubt there are many more not yet discovered. But if one seriously wished to search for an $r = 5$ one would be well advised to go beyond L_5.

A case of $s = 4$ probably occurs for a smaller D, but the method in [2] is particularly ineffective for its discovery since this method is particularly inclined toward the escalatory case. The statistics show that: 132 cases out of 132! If one seriously wished to search for a $Q(\sqrt{D})$ with $s = 4$ and a modest D (say, about 10^{11} or 10^{12}), there are probably better methods. The Complementary Series 3 and 6 of [10] *always* have $r = s$, and this could be easily mechanized. Every K' not having $81 \mid h$ could be discarded as in phase II above. In fact, during the computation of its regulator R, the computation can be terminated as soon as it is apparent that its $h < 81$. That will occur frequently.

The new development [3] referred to in Section 1 is also based upon a series of d that have $r = s$. Craig [11] has recently shown how to construct infinitely many $Q(\sqrt{d})$ with $r \geqslant 4$. The smallest known of these, which has $d \approx -428 \cdot 10^{100}$, actually was the first $Q(\sqrt{d})$ with $r \geqslant 4$ ever discovered; see [12], [7]. It is shown in [3] that this K has $r = s$, so its K' has $s \geqslant 4$. Here is its D:

$$\begin{array}{rrrr} D = 1284 & 0625510361 & 2492395282 & 3484951333 \\ 3657649481 & 0472771825 & 7285040631 & 6022716187 \\ 3462515321 & 3764715019 & 5799772957. & \end{array}$$

We have $D = 3 \cdot 83 \cdot 239 \cdot 50503 \cdot 262151 \cdot 586057 \cdot 2824139 \cdot 6607829x$, where the 70 digit number x has not been factored. We know that x is composite and that every prime factor of x must exceed 2×10^9, but we do not know if x is square-free. If $D = B^2\delta$ with δ square-free, then $K' = Q(\sqrt{\delta})$ has $s \geqslant 4$. Note the vagueness; for all we know it could have $s = 6$. It remains desirable to have a $Q(\sqrt{D})$ with $s = 4$ and a much smaller D.

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