

Semidiscretization in Time for Parabolic Problems

By Marie-Noëlle Le Roux

Abstract. We study the error to the discretization in time of a parabolic evolution equation by a single-step method or by a multistep method when the initial condition is not regular.

Introduction. The problem we are considering is the parabolic evolution equation

$$(*) \quad \begin{cases} u'(t) + Au(t) = 0, & 0 < t \leq T, \\ u(0) = u_0. \end{cases}$$

Here, A is a linear operator, unbounded on Hilbert space H , of domain $D(A)$ dense in H ; the initial value u_0 is assumed to be only in H .

In the first part, we study the error due to the discretization in time of the problem (*) by a single-step method. The scheme is defined by the choice of a rational approximation $r(z)$ to the exponential e^{-z} for complex variable z . For the case of A selfadjoint, these methods are analyzed in [1] and [2]. Also in the special case of one space dimension, similar results can be found in [9]. For the case of A non-selfadjoint, the result for the special case $r(z) = 1/(1+z)$ was obtained by Blair [3] and by Fujita and Mizutani [6]. Using the technique in [1], we generalize these results when the method is strongly $A(\theta)$ -stable ($0 < \theta \leq \pi/2$). Concerning examples, a class of rational approximations $\{r_p(z)\}$ to e^{-z} which are strongly $A(0)$ -stable with $p \geq 3$ is documented in [8] and [2]. It is shown in [8] that for $p \geq 3$, r_p is in fact strongly $A(\theta_p)$ -stable for some $0 < \theta_p < \pi/2$. For small p , θ_p is close to $\pi/2$ and in the special cases $p = 3, 4$, r_p is A -stable. Examples of rational approximations to e^{-z} which are strongly $A(\theta)$ -stable with $r(\infty) = 0$ are provided by the family $r_\nu(z)$ developed in [2].

In the second part, we investigate error estimates when the discretization in time is carried out by means of a multistep method. Zlamal gives an error bound under the assumption that the operator A is selfadjoint and the method strongly $A(0)$ -stable. Here, error estimates are obtained if the operator A is maximal sectorial and the method strongly $A(\theta)$ -stable ($0 \leq \theta \leq \pi/2$).

I. Semidiscretization in Time by a Single-Step Method.

1. *Introduction.* Let A be a linear operator, unbounded on Hilbert space H ,

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of domain $D(A)$ dense in H . A is supposed to be maximal sectorial [7]; and we consider the problem

$$(1) \quad \begin{cases} u'(t) + Au(t) = 0, & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

where u_0 belongs to H . The approximate values u_n of u at the time level $t_n = n\Delta t$, (Δt denotes the time increment) are determined by

$$(2) \quad u_{n+1} = r(\Delta t A)u_n, \quad n \geq 0,$$

where $r(z)$ is a rational function of the complex variable z defining the single-step method. The error at the time t_n is given by

$$|u_n - u(t_n)|_H = |(e^{-t_n A} - r^n(\Delta t A))u_0|_H;$$

hence, we deduce

$$(3) \quad |u_n - u(t_n)|_H \leq \|e^{-t_n A} - r^n(\Delta t A)\|_{\mathfrak{L}(H,H)} |u_0|_H.$$

2. *Assumptions on the Method.* We assume that the single-step method is of the order p ($p \geq 1$); then there are two constants $\sigma_1 > 0$ and $C > 0$ such that

$$|e^{-z} - r(z)| \leq C|z|^{p+1}, \quad \forall z \in \mathbb{C}, |z| \leq \sigma_1.$$

LEMMA 1.1. *Let the single-step method be of the order p , then, for any $\theta \in [0, \pi/2]$, there are constants σ, β and $c > 0$ depending only on r and θ such that for $z \in \mathbb{C}, |z| \leq \sigma, -\theta \leq \text{Arg } z \leq +\theta$,*

$$(4) \quad |r^n(z) - e^{-nz}| \leq C_n |z|^{p+1} e^{-\beta n \text{Re } z}.$$

Proof. We have the equality

$$|r^n(z) - e^{-nz}| = |r(z) - e^{-z}| \left| \sum_{j=0}^{n-1} r^j(z) e^{-(n-1-j)z} \right|.$$

Since the method is assumed of the order p , we have

$$|r(z) - e^{-z}| \leq C|z|^{p+1}$$

and,

$$|r(z)| \leq e^{-\text{Re } z} (1 + C|z|^{p+1}), \quad \forall z \in \mathbb{C}, |z| \leq \sigma_1.$$

Let $z \in \mathbb{C}$ such that $-\theta \leq \text{Arg } z \leq +\theta, \theta \in [0, \pi/2]$, then, $\text{Re } z \geq |z| \cos \theta$; hence

$$|r(z)| \leq e^{-\frac{1}{2} \text{Re } z} (e^{-\frac{1}{2} |z| \cos \theta} + C|z|^{p+1})$$

and there is a constant $\sigma \leq \sigma_1$ such that

$$C|z|^{p+1} \leq \frac{1}{2} |z| \cos \theta, \quad |z| \leq \sigma.$$

Therefore,

$$|r(z)| \leq e^{-\frac{1}{2} \text{Re } z} \quad \text{for } |z| \leq \sigma, -\theta \leq \text{Arg } z \leq +\theta$$

and

$$|r^n(z) - e^{-nz}| \leq C|z|^{p+1} \sum_{j=0}^{n-1} e^{-j \operatorname{Re} z/2} e^{-(n-1-j)\operatorname{Re} z}.$$

Hence,

$$|r^n(z) - e^{-nz}| \leq Cn|z|^{p+1} e^{-n \operatorname{Re} z/2}.$$

This concludes the proof.

We also assume that the method is strongly $A(\theta)$ -stable, i.e. if θ is not zero, $\forall z \in S_\theta$, $|r(z)| \leq 1$ and $|r(\infty)| < 1$, where S_θ is the sector $\{z \in \bar{\mathbb{C}}/z = \infty \text{ or } 0 \text{ or } -\theta \leq \operatorname{Arg} z \leq \theta\}$; if $\theta = 0$, $\forall x > 0$, $|r(x)| < 1$ and $|r(\infty)| < 1$.

3. Error Estimates in the Case of a Selfadjoint Positive Operator.

THEOREM 1.1. *Let A be a selfadjoint positive operator and let the single-step method be strongly $A(0)$ -stable and of the order p . Then, there is a constant C depending only on the method such that,*

$$(5) \quad |u_n - u(t_n)|_H \leq C \frac{\Delta t^p}{t_n^p} |u_0|_H \quad \text{for } n \geq 1.$$

Proof. Since the operator A is selfadjoint, we have

$$\|e^{-t_n A} - r^n(\Delta t A)\|_{\mathfrak{L}(H,H)} = \sup_{z \in S_p(A)} |e^{-t_n z} - r^n(\Delta t z)| \leq \sup_{x \geq 0} |e^{-nx} - r^n(x)|.$$

Let $x \in \mathbb{R}_+$ such that $x \leq \sigma$ (σ has the value defined in the Lemma 1.1); we get

$$|e^{-nx} - r^n(x)| \leq Cnx^{p+1} e^{-\beta nx} \leq \frac{C}{n^p} = C \frac{\Delta t^p}{t_n^p}.$$

Let $x \in \mathbb{R}_+$ such that $x \geq \sigma$; then

$$|e^{-nx} - r^n(x)| \leq e^{-n\sigma} + \sup_{x \geq \sigma} |r(x)|^n.$$

Since the method is strongly $A(0)$ -stable,

$$\sup_{x \geq \sigma} |r(x)| = r < 1;$$

hence,

$$|e^{-nx} - r^n(x)| \leq e^{-n\sigma} + r^n \leq \frac{C}{n^p} = C \frac{\Delta t^p}{t_n^p}.$$

Then we get

$$\|e^{-t_n A} - r^n(\Delta t A)\|_{\mathfrak{L}(H,H)} \leq C \frac{\Delta t^p}{t_n^p},$$

and the result follows from (3).

4. *Error Estimates When A is Not a Selfadjoint Operator.* In this case, we shall need the following lemma:

LEMMA 1.2. *Let A be a maximal positive operator for which there is some constant θ_0 ($0 \leq \theta_0 \leq \pi/2$) such that*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Let φ be a continuous function on the sector S_θ ($\theta_0 < \theta < \pi/2$) which is holomorphic in the interior of S_θ and satisfies for some constant $R > 0$ and two functions f_1 and f_2 from \mathbf{R}_+ to \mathbf{R}_+ the following estimates:

$$(6) \quad \forall z \in S_\theta, |z| \leq R, \quad |\varphi(z)| \leq f_1(|z|),$$

$$(7) \quad \forall z \in S_\theta, |z| \geq R, \quad |\varphi(z) - \varphi(\infty)| \leq f_2(|z|).$$

Further, assume that the functions f_1, f_2 satisfy

$$\int_0^R f_1(r) \frac{dr}{r} < +\infty \quad \text{and} \quad \int_R^{+\infty} f_2(r) \frac{dr}{r} < +\infty.$$

Then there is a constant C such that

$$(8) \quad \|\varphi(A)\|_{\mathfrak{L}(H,H)} \leq \frac{C}{\theta - \theta_0} \left\{ \int_0^R f_1(r) \frac{dr}{r} + \int_R^{+\infty} f_2(r) \frac{dr}{r} + \left(R + \frac{1}{R}\right) |\varphi(\infty)| \right\} + |\varphi(\infty)|.$$

Proof. We set

$$h(z) = \varphi(z) - \frac{z}{1+z} \varphi(\infty).$$

We have

$$\varphi(A) = h(A) + \varphi(\infty)A(I+A)^{-1} \quad \text{and} \quad \|A(I+A)^{-1}\|_{\mathfrak{L}(H,H)} \leq 1.$$

Hence,

$$\|\varphi(A)\|_{\mathfrak{L}(H,H)} \leq \|h(A)\|_{\mathfrak{L}(H,H)} + |\varphi(\infty)|.$$

Besides, we have

$$h(A) = \frac{1}{2\pi i} \int_{\Gamma} h(z)(zI - A)^{-1} dz,$$

where Γ is the continuous, positively oriented curve defined by $\text{Arg } z = \pm \theta$. Let $\Gamma_1 = \{z \in \Gamma, |z| \leq R\}$ and $\Gamma_2 = \{z \in \Gamma, |z| \geq R\}$. For $z \in \Gamma$, the following estimate holds [5]

$$\|(zI - A)^{-1}\|_{\mathfrak{L}(H,H)} \leq \frac{C}{\theta - \theta_0} \frac{1}{|z|}.$$

Now, from (6), we get

$$|h(z)| = \left| \varphi(z) - \frac{z}{1+z} \varphi(\infty) \right| \leq f_1(|z|) + |z| |\varphi(\infty)|, \quad \forall z \in \Gamma_1;$$

hence

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_1} h(z)(zI - A)^{-1} dz \right\| \leq \frac{C}{\theta - \theta_0} \left(\int_0^R f_1(r) \frac{dr}{r} + R |\varphi(\infty)| \right);$$

from (7), we get

$$|h(z)| = \left| \varphi(z) - \varphi(\infty) + \frac{1}{1+z} \varphi(\infty) \right| \leq f_2(|z|) + \frac{1}{|z|} |\varphi(\infty)|, \quad \forall z \in \Gamma_2;$$

hence,

$$\frac{1}{2\pi i} \int_{\Gamma_2} h(z)(zI - A)^{-1} dz \leq \frac{C}{\theta - \theta_0} \left(\int_R^\infty f_2(r) \frac{dr}{r} + \frac{1}{R} |\varphi(\infty)| \right).$$

The estimate (8) now follows immediately.

THEOREM 1.2. *Let A be a maximal positive operator satisfying for some constant θ_0 ($0 \leq \theta_0 < \pi/2$),*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Further, assume that the single-step method is of the order p and strongly $A(\theta)$ -stable ($\theta_0 < \theta < \pi/2$). Then there is a constant C depending only on the single-step method θ and θ_0 such that

$$(9) \quad |u_n - u(t_n)|_H \leq C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

Proof. We apply Lemma 1.2 with $\varphi(z) = e^{-nz} - r^n(z)$. Then, from (4), we get

$$\forall z \in S_\theta, |z| \leq \sigma, \quad |\varphi(z)| \leq Cn |z|^{p+1} e^{-\beta n |z| \cos \theta}$$

and

$$\int_0^{+\infty} nr^p e^{-\beta nr \cos \theta} dr \leq \frac{C}{n^p} \int_0^\infty x^p e^{-x} dx \leq C' \frac{\Delta t^p}{t_n^p}.$$

Besides, we have

$$\forall z \in S_\theta, |z| \geq \sigma, \quad |\varphi(z) - \varphi(\infty)| \leq e^{-n|z| \cos \theta} + |r^n(z) - r^n(\infty)|$$

and

$$r^n(z) - r^n(\infty) = (r(z) - r(\infty)) \sum_{j=0}^{n-1} r^j(z) r^{n-1-j}(\infty).$$

Since the method is strongly $A(\theta)$ -stable, we may set

$$\sup_{z \in S_\theta; |z| \geq \sigma} |r(z)| = e^{-\delta}$$

for some $\delta > 0$; and since r is a rational function, there is a constant C such that

$$|r(z) - r(\infty)| \leq \frac{C}{|z|} \text{ for } |z| \geq \sigma.$$

Hence, for $|z| \geq \sigma$

$$|\varphi(z) - \varphi(\infty)| \leq e^{-n|z|\cos\theta} + \frac{C}{|z|} e^{-n\delta}$$

and

$$\int_\sigma^{+\infty} \left(\frac{e^{-nr\cos\theta}}{r} + C \frac{e^{-n\delta}}{r^2} \right) dr \leq C \left(\frac{1}{n^p} + e^{-n\delta} \right) \leq C' \frac{\Delta t^p}{t_n^p}.$$

Besides, $|\varphi(\infty)| = |r^n(\infty)|$ and $|r^n(\infty)| \leq C/n^p$, since $|r(\infty)| < 1$; then using (8), (9) follows.

II. Semidiscretization in Time by a Multistep Method.

1. *Introduction.* We again consider the equation (1). Let ρ and σ be two real polynomials of degree less than or equal to q ,

$$\rho(\zeta) = \sum_{i=0}^q \alpha_i \zeta^i \text{ and } \sigma(\zeta) = \sum_{i=0}^q \beta_i \zeta^i \quad (\alpha_q > 0).$$

The approximate values u_n of u at the time level $t_n = n\Delta t$ are determined by

$$(10) \quad \sum_{i=0}^q (\alpha_i + \Delta t \beta_i A) u_{n+i} = 0,$$

assuming the starting values u_0, u_1, \dots, u_{q-1} to be given (by another method).

2. *Assumptions on the Method.* (a) We assume that the multistep method is of order p ; then we have

$$(11) \quad \sum_{i=0}^q i^l \alpha_i = l \sum_{i=0}^q i^{l-1} \beta_i, \quad l = 0, 1, \dots, p.$$

(b) We also assume that the (ρ, σ) method is strongly $A(\theta)$ -stable. We set $\bar{\omega}(\zeta; z) = \rho(\zeta) + z\sigma(\zeta)$,

$$S_\theta = \{z \in \bar{\mathbb{C}}/z = \infty \text{ or } z = 0 \text{ or } -\theta \leq \text{Arg } z \leq +\theta\}.$$

The method is strongly $A(\theta)$ -stable ($0 < \theta \leq \pi/2$) if and only if the modulus of all roots of the polynomials $\bar{\omega}(\cdot, z)$ are less than one for any z in the interior of S_θ . If $\theta = 0$, the method is strongly $A(0)$ -stable if and only if for any $x > 0$ the modulus of all roots of the polynomials $\bar{\omega}(\cdot, x)$ and σ are less than one; the roots of the polynomial ρ with modulus equal to one, ζ_i , are simple and the growth parameters λ_i satisfy $\text{Re } \lambda_i > 0$; these growth parameters λ_i are given by

$$\lambda_i = \sigma(\zeta_i) / \zeta_i \rho'(\zeta_i).$$

We now define

$$\begin{aligned} \delta_i(z) &= \frac{\alpha_i + \beta_i z}{\alpha_q + \beta_q z} \quad (0 \leq i \leq q), \\ \gamma_l(z) &= 0 \quad \text{for } l < 0, \\ \gamma_0(z) &= 1, \\ \sum_{k=0}^q \gamma_{l-k}(z) \delta_{q-k}(z) &= 0 \quad \text{for } l > 0, l \in \mathbf{Z}, \\ E_l(z) &= \sum_{i=0}^q \delta_i(z) e^{-(l+i)z}, \quad l \geq 0. \end{aligned}$$

Then, we have

$$\sum_{i=0}^q \delta_i(\Delta t A) (u(t_{n+i}) - u_{n+i}) = E_n(\Delta t A) u_0.$$

Hence [5]

$$\begin{aligned} u(t_{n+q}) - u_{n+q} &= \sum_{s=0}^{q-1} \sum_{k=0}^s \gamma_{n-k}(\Delta t A) \delta_{s-k}(\Delta t A) [u_s - u(t_s)] \\ &\quad + \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) u_0. \end{aligned}$$

LEMMA 2.1. *Let the method (ρ, σ) be of the order p and $A(\theta)$ -stable; then for any $z \in \mathbf{C}$ with $\text{Re } z \geq 0$, we have*

$$(12) \quad |E_l(z)| \leq C |z|^{p+1} e^{-l \text{Re } z}, \quad l \geq 0.$$

Proof. We set

$$v(t) = e^{-tz}, \quad t \geq 0, \text{Re } z \geq 0.$$

We have

$$E_l(z) = (\alpha_q + \beta_q z)^{-1} \left\{ \sum_{i=0}^q \alpha_i v(l+i) - \sum_{i=0}^q \beta_i v'(l+i) \right\}.$$

Since the method is of the order p , we get

$$\begin{aligned} E_l(z) &= (\alpha_q + \beta_q z)^{-1} \left\{ \sum_{i=1}^q \alpha_i \int_l^{l+i} \frac{(l+i-t)^p}{p!} v^{(p+1)}(t) dt \right. \\ &\quad \left. - \sum_{i=0}^q \beta_i \int_l^{l+i} \frac{(l+i-t)^{p-1}}{(p-1)!} v^{(p+1)}(t) dt \right\}. \end{aligned}$$

Now, $v^{(p+1)}(t) = (-z)^{p+1} e^{-tz}$ and, since the method is strongly $A(\theta)$ -stable, we have $\alpha_q \beta_q > 0$. Hence,

$$|E_l(z)| \leq \frac{1}{|\alpha_q|} \left(\sum_{i=1}^q \frac{|\alpha_i|}{p!} i^p + \sum_{i=0}^q \frac{|\beta_i|}{(p-1)!} i^{p-1} \right) |z|^{p+1} e^{-l \operatorname{Re} z}.$$

LEMMA 2.2. *Let the method (ρ, σ) be strongly $A(\theta)$ -stable $(0 \leq \theta \leq \pi/2)$. Let $z \in \mathbb{C}$ such that $-\theta_1 \leq \operatorname{Arg} z \leq \theta_1$ $(0 \leq \theta_1 < \theta)$ if $\theta > 0$ and $z \in \mathbb{R}, z \geq 0$, if $\theta = 0$. Then, there are constants R, μ, C , such that*

$$(13) \quad \begin{cases} |\gamma_l(z)| \leq Ce^{-\mu l} & \text{for } |z| \geq R, \\ |\gamma_l(z)| \leq Ce^{-\mu l |z|} & \text{for } |z| \leq R. \end{cases}$$

Proof. The first inequality follows from the following result [5]. If the method (ρ, σ) is strongly $A(\theta)$ -stable, there are constants $c \in]0, 1[$, $R > 0$ and $c_\infty > 0$ such that

$$\forall z \in S_\theta, |z| \geq R, \quad |\gamma_l(z)| \leq c_\infty c^l.$$

The second inequality follows from the following result [5]. If the method (ρ, σ) is strongly $A(\theta)$ -stable $(0 \leq \theta \leq \pi/2)$, for any $a > 0$, there are two constants μ' and $C > 0$ such that, for any $x \in \mathbb{R}_+, x \leq a$ and for any $\lambda \in \Gamma_\theta$, $(\Gamma_\theta = \{z \in \mathbb{C} / z = \infty \text{ or } \operatorname{Arg} z = \pm \theta\})$, $|\gamma_l(x + \lambda)| \leq Ce^{-\mu' l x}$.

Case (i); $\theta > 0$. Let $z \in \mathbb{C}$ such that $|z| \leq R, \operatorname{Arg} z = \alpha, 0 \leq \alpha \leq \theta_1$; then we have

$$z = |z| \frac{\sin(\theta - \alpha)}{\sin \theta} + |z| \frac{\sin \alpha}{\sin \theta} e^{i\theta},$$

and

$$|z| \frac{\sin(\theta - \alpha)}{\sin \theta} \leq R.$$

Hence,

$$|\gamma_l(z)| \leq Ce^{-\mu' l |z| \sin(\theta - \alpha) / \sin \theta}$$

that is,

$$|\gamma_l(z)| \leq Ce^{-\mu l |z|} \quad \text{with} \quad \mu = \mu' \frac{\sin(\theta - \theta_1)}{\sin \theta}.$$

Case (ii); $\theta = 0$. Let $x \leq R$; then there are two constants μ' and C such that $|\gamma_l(x)| \leq Ce^{-\mu' l x}$ and (13) follows.

3. Error Estimates in the Case of a Selfadjoint Positive Operator.

THEOREM 2.1. *Let A be a selfadjoint positive operator and let the (ρ, σ) method be strongly $A(0)$ -stable and of the order p . Further, assume that the starting values are obtained by a single-step weakly $A(0)$ -stable method of the order $p - 1$; then there is a constant C depending only on the (ρ, σ) method and on the single-step method such that*

$$(14) \quad |u_n - u(t_n)|_H \leq C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

Proof. Since the operator A is selfadjoint and positive, we have

$$\left\| \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leq \sup_{x \geq 0} \left| \sum_{l=0}^n \gamma_{n-l}(x) E_l(x) \right|.$$

Let $x \in \mathbf{R}_+$, such that $x \leq R$ (R is defined in Lemma 2.2).

From Lemmas 2.1 and 2.2, we get

$$|\gamma_{n-l}(x)| \leq C e^{-(n-l)x}, \quad 0 \leq l \leq n,$$

and

$$|E_l(x)| \leq C x^{p+1} e^{-lx}, \quad l \geq 0.$$

Set $\nu = \inf(\mu, 1)$; then

$$\left| \sum_{l=0}^n \gamma_{n-l}(x) E_l(x) \right| \leq C n e^{-\nu n x} x^{p+1} \leq \frac{C}{n^p}.$$

Now, let $x \geq R$ and we have the estimates

$$|\gamma_{n-l}(x)| \leq C e^{-\mu(n-l)}, \quad 0 \leq l \leq n,$$

$$|E_l(x)| \leq C x^{p+1} e^{-lx}, \quad 1 \leq l \leq n,$$

$$|E_0(x)| \leq C.$$

Hence,

$$\left| \sum_{l=0}^n \gamma_{n-l}(x) E_l(x) \right| \leq C \left\{ e^{-\mu n} + \sum_{l=1}^n e^{-\mu(n-l)} e^{-lx} x^{p+1} \right\}.$$

Set $\nu = \inf(\mu, R)$; then

$$\left| \sum_{l=0}^n \gamma_{n-l}(x) E_l(x) \right| \leq C \{ e^{-\mu n} + n e^{-\nu n} e^{-(x-\nu)} x^{p+1} \} \leq \frac{C}{n^p}.$$

Hence,

$$\left\| \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leq \frac{C}{n^p}.$$

Since we have assumed that the starting values are obtained by a weakly $A(0)$ -stable single-step method of order $p - 1$,

$$u_s = r^s(\Delta t A) u_0, \quad 0 \leq s \leq q - 1,$$

where r is a rational function satisfying $\forall x \geq 0, |r(x)| \leq 1$ and for which there are constants $\sigma > 0$ and $c > 0$ such that

$$|r(x) - e^{-x}| \leq C |x|^p \quad \forall x \leq \sigma.$$

Since the operator A is selfadjoint and positive, we have

$$\| \gamma_{n-k}(\Delta t A) \delta_{s-k}(\Delta t A) (u_s - u(t_s)) \| \leq \sup_{x \geq 0} | \gamma_{n-k}(x) \delta_{s-k}(x) (r^s(x) - e^{-sx}) \| u_0 \|_H.$$

Let $x \in \mathbf{R}_+$, $x \leq \inf(\sigma, R)$; then

$$|\gamma_{n-k}(x)| \leq Ce^{-\mu(n-k)x} \leq Ce^{-\mu nx} e^{\mu qx}, \quad |\delta_{s-k}(x)| \leq C,$$

and

$$|r^s(x) - e^{-sx}| \leq |r(x) - e^{-x}| \left| \sum_{j=0}^{s-1} r^j(x) e^{-(s-1-j)x} \right| \leq Cx^p.$$

Hence,

$$|\gamma_{n-k}(x)\delta_{s-k}(x)(r^s(x) - e^{-sx})| \leq Ce^{-\mu nx} x^p \leq \frac{C'}{n^p} = C' \frac{\Delta t^p}{t_n^p}.$$

Let $x \geq \inf(\sigma, R)$; then

$$|\gamma_{n-k}(x)| \leq Ce^{-\mu(n-k)} \leq Ce^{\mu q} e^{-\mu n},$$

so that

$$|\gamma_{n-k}(x)\delta_{s-k}(x)(r^s(x) - e^{-sx})| \leq Ce^{-\mu n} \leq \frac{C'}{n^p} = C' \frac{\Delta t^p}{t_n^p}.$$

Hence,

$$\left\| \sum_{s=0}^{q-1} \sum_{k=0}^s \gamma_{n-k}(\Delta t A) \delta_{s-k}(\Delta t A) (u_s - u(t_s)) \right\| \leq C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

4. Error Estimates When A is Not a Selfadjoint Operator.

THEOREM 2.2. *Let A be a maximal positive operator for which there is some constant θ_0 ($0 \leq \theta_0 < \pi/2$) such that*

$$\forall u \in D(A), \quad (Au, u) \in S_{\theta_0}.$$

Let the (ρ, σ) method be strongly $A(\theta)$ -stable ($\theta_0 < \theta < \pi/2$) and of the order p . Further, assume that the starting values are obtained by a weakly $A(\theta)$ -stable, single-step method of order $p-1$; then there is a constant C depending only on the (ρ, σ) method, on the single-step method, on θ and θ_0 such that

$$(15) \quad |u_n - u(t_n)|_H \leq C \frac{\Delta t^p}{t_n^p} |u_0|_H.$$

Proof. We apply Lemma 1.2 to estimate

$$\left\| \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{F}(H, H)}.$$

From Lemmas 2.1 and 2.2, for any $z \in S_{\theta_1}$ and $|z| \leq R$ ($\theta_1 \in]\theta_0, \theta[$), we have

$$\sum_{l=0}^n \gamma_{n-l}(z) E_l(z) \leq C \sum_{l=0}^n e^{-\mu(n-l)|z|} |z|^{\rho+1} e^{-l|z| \cos \theta_1}.$$

Set $\nu = \inf(\mu, \cos \theta_1)$; then

$$\left| \sum_{l=0}^n \gamma_{n-l}(z)E_l(z) \right| \leq Cne^{-\nu n|z|}|z|^{p+1}$$

and

$$\int_0^\infty ne^{-\nu nr}r^p dr = \frac{1}{n^p} \left(\int_0^\infty e^{-\nu\rho}\rho^p d\rho \right) \leq \frac{C}{n^p}.$$

Besides, for $l \geq 1$, $E_l(\infty) = 0$ and from (13) and (14), for $z \in S_{\theta_1}$, $|z| \geq R$

$$\left| \sum_{l=1}^n \gamma_{n-l}(z)E_l(z) \right| \leq C \sum_{l=1}^n e^{-\mu(n-l)}|z|^{p+1}e^{-l|z|\cos\theta_1}.$$

Now, set $\nu = \inf(\mu, R \cos \theta_1)$; then

$$e^{-\mu(n-l)}e^{-\nu|z|\cos\theta_1} \leq e^{-\nu n}e^{-l(|z|\cos\theta_1-\nu)}$$

and

$$\left| \sum_{l=1}^n \gamma_{n-l}(z)E_l(z) \right| \leq Cne^{-\nu n}e^{\nu-|z|\cos\theta_1}|z|^{p+1}$$

and

$$\int_0^\infty ne^{-\nu n}e^{\nu-r\cos\theta_1}r^p dr \leq \frac{C}{n^p}.$$

For $l = 0$, $E_0(\infty)$ may be not equal to zero,

$$\lim_{|z| \rightarrow \infty; z \in S_\theta} E_0(z) = \frac{\beta_0}{\beta_q}.$$

We have

$$\left| \gamma_n(z)E_0(z) - \gamma_n(\infty)\frac{\beta_0}{\beta_q} \right| \leq |\gamma_n(z) - \gamma_n(\infty)||E_0(z)| + |E_0(z) - E_0(\infty)||\gamma_n(\infty)|.$$

Now, for $|z| \geq R$, we have [5]

$$|\gamma_n(z) - \gamma_n(\infty)| \leq C \frac{e^{-\mu n}}{1 + (\text{Log}(1 + |z|))^2},$$

and $|E_0(z)| \leq C$. Also,

$$E_0(z) - E_0(\infty) = \frac{\alpha_0\beta_q - \alpha_q\beta_0}{\alpha_q + \beta_q z} + \sum_{i=1}^q \delta_i(z)e^{-iz};$$

hence, for $z \in S_{\theta_1}$ and $|z| \geq R$,

$$|E_0(z) - E_0(\infty)| \leq \frac{C}{|z|} + Ce^{-|z|\cos\theta_1},$$

Then, since $|\gamma_n(\infty)| \leq Ce^{-\mu n}$, we get

$$\left| \gamma_n(z)E_0(z) - \gamma_n(\infty) \frac{\beta_0}{\beta_q} \right| \leq Ce^{-\mu n} \left\{ \frac{1}{1 + (\text{Log}(1 + |z|))^2} + \frac{1}{|z|} + e^{-|z| \cos \theta_1} \right\}$$

and

$$\int_R^{+\infty} e^{-\mu n} \left(\frac{1}{r(1 + \text{Log}(1 + r))^2} + \frac{1}{r^2} + \frac{e^{-r \cos \theta_1}}{r} \right) dr \leq Ce^{-\mu n}.$$

Then, from Lemma 1.2, we get

$$\left\| \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leq C \left\{ \frac{1}{n^p} + e^{-\mu n} + |\gamma_n(\infty)E_0(\infty)| \right\};$$

hence,

$$\left\| \sum_{l=0}^n \gamma_{n-l}(\Delta t A) E_l(\Delta t A) \right\|_{\mathfrak{L}(H,H)} \leq \frac{C}{n^p}.$$

Now, we assume that the starting values are obtained by a weakly $A(\theta)$ -stable single-step method, of the order $p - 1$; then

$$u_s = r^s(\Delta t A)u_0, \quad 0 \leq s \leq q - 1,$$

where r is a rational function satisfying $\forall z \in S_\theta, |r(z)| \leq 1$ and for which there are some constants σ and C such that

$$|r(z) - e^{-z}| \leq C|z|^p \quad \forall z \in S_\theta, |z| \leq \sigma.$$

We again apply Lemma 1.2 to estimate

$$\| \gamma_{n-k}(\Delta t A) \delta_{s-k}(\Delta t A) (r^s(\Delta t A) - e^{-s\Delta t A}) \|_{\mathfrak{L}(H,H)}, \quad 0 \leq k \leq s, 0 \leq s \leq q - 1.$$

Set $\sigma_1 = \inf(\sigma, R)$. Then for $z \in S_{\theta_1}, |z| \leq \sigma_1$ and $n > k$,

$$|\gamma_{n-k}(z) \delta_{s-k}(z) (r^s(z) - e^{-sz})| \leq Ce^{-\mu n} |z| |z|^p$$

and

$$\int_0^{+\infty} e^{-\mu n r} r^{p-1} dr \leq \frac{C}{n^p}.$$

Besides, we have

$$\begin{aligned} & |\gamma_{n-k}(z) \delta_{s-k}(z) (r^s(z) - e^{-sz}) - \gamma_{n-k}(\infty) \delta_{s-k}(\infty) r^s(\infty)| \\ & \leq |\gamma_{n-k}(z) - \gamma_{n-k}(\infty)| |\delta_{s-k}(z) r^s(z)| + |\gamma_{n-k}(\infty)| |\delta_{s-k}(z) - \delta_{s-k}(\infty)| |r^s(z)| \\ & \quad + |\gamma_{n-k}(\infty)| |\delta_{s-k}(\infty)| |r^s(z) - r^s(\infty)|. \end{aligned}$$

Now, for $|z| \geq \sigma_1$ we have

$$|\gamma_{n-k}(z) - \gamma_{n-k}(\infty)| \leq C \frac{e^{-\mu n}}{1 + (\text{Log}(1 + |z|))^2};$$

and since r and δ_{s-k} are rational functions,

$$|\delta_{s-k}(z) - \delta_{s-k}(\infty)| \leq \frac{C}{|z|}, \quad |r(z) - r(\infty)| \leq \frac{C}{|z|} \quad \text{for } |z| \geq \sigma_1.$$

Also,

$$|\gamma_n(\infty)| \leq Ce^{-\mu n} \quad \text{and} \quad |\delta_{s-k}(z)| \leq C \quad \text{for } |z| \geq \sigma_1.$$

Hence,

$$\begin{aligned} & |\gamma_{n-k}(z)\delta_{s-k}(z)(r^s(z) - e^{-sz}) - \gamma_{n-k}(\infty)\delta_{s-k}(\infty)r^s(\infty)| \\ & \leq Ce^{-\mu n} \left(\frac{1}{1 + (\text{Log}(1 + |z|))^2} + \frac{1}{|z|} \right), \end{aligned}$$

and from Lemma 1.2, it follows that

$$\begin{aligned} & \|\gamma_{n-k}(\Delta t A)\delta_{s-k}(\Delta t A)(r^s(\Delta t A) - e^{-s\Delta t A})\| \\ & \leq C \left\{ \frac{1}{n^p} + |\gamma_{n-k}(\infty)\delta_{s-k}(\infty)r^s(\infty)| \right\} \leq \frac{C''}{n^p}. \end{aligned}$$

Université de Rennes
U.E.R. Mathématiques et Informatique
Rennes-Beaulieu 35042 Rennes Cedex
France

1. G. A. BAKER & J. H. BRAMBLE, "The construction and analysis of high order single-step Galerkin approximation for parabolic equations." (Preprint.)
2. G. A. BAKER, J. H. BRAMBLE & V. THOMÉE, "Single step Galerkin approximations for parabolic problems," *Math. Comp.*, v. 31, 1977, pp. 818–847.
3. J. BLAIR, *Approximate Solution of Elliptic and Parabolic Boundary Value Problems*, Thesis, University of California, Berkeley, 1970.
4. M. CROUZEIX, *Sur l'Approximation des Équations Différentielles Opérationnelles par des Méthodes de Runge-Kutta*, Thesis, University of Paris VI, 1975.
5. M. CROUZEIX & P. A. RAVIART, "Approximation d'équations d'évolution linéaires par des méthodes multi-pas." (To appear.)
6. H. FUJITA & A. MIZUTANI, "On the finite element method for parabolic equations. I: Approximation of holomorphic semi-groups," *J. Math. Soc. Japan*, v. 28, 1976, pp. 749–771.
7. T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1966.
8. S. P. NORSETT, "One-step methods of Hermite type for numerical integration of stiff systems," *BIT*, v. 14, 1974, pp. 63–77.
9. V. THOMÉE, "Some convergence results for Galerkin methods for parabolic boundary value problems," *Mathematical Aspects of Finite Elements in Partial Differential Equations* (Carl de Boor, Ed.), Academic Press, New York, 1974, pp. 55–88.
10. M. ZLAMAL, *Finite Element Multistep Methods for Parabolic Equations*, ISNM 28, Birkhäuser Verlag, Basel and Stuttgart, 1975.
11. M. ZLAMAL, "Finite element multistep discretizations of parabolic boundary value problems," *Math. Comp.*, v. 29, 1975, pp. 350–359.