Cyclic-Sixteen Class Fields for $\mathbb{Q}(-p)^{1/2}$ by Modular Arithmetic

By Harvey Cohn*

Abstract. Numerical experiments result in the construction of cyclic-sixteen class fields for $\mathbb{Q}(-p)^{1/2}$, $p$ prime < 2000, by radicals involving quadratic and biquadratic parameters. These fields are characterized by rational factorization properties modulo a variable prime; but it suffices to use only three primes selected and checked by computer to verify the class field, if earlier work (jointly with Cooke) on the cyclic-eight class field is utilized.

1. Introduction. To give a specific example of a new result in rational arithmetic, the current computation shows that a (large) prime $q$ satisfies $q = x^2 + 257y^2$ (in $\mathbb{Z}$) exactly when a certain equation over $\mathbb{Q}$ of degree 32 splits into 32 (different) linear factors modulo $q$. The general root of this equation is expressible (with "too many conjugates") as $\Lambda_0^{1/2}$, where

$$\Lambda_0 = (-5 + 2(-257)^{1/2})(1 + (1 + 16i)^{1/2})$$

$$\cdot \left( \frac{-9 + (-257)^{1/2}}{1 - i} \right) \left( 16 + 257^{1/2} \right)^{1/2},$$

so that the radicals in $\Lambda_0$ must be chosen with correct signs. It will prove advantageous to replace a rather appalling equation of degree 32 by the following system of five quadratic congruences in which the signs are implicitly specified:

$$\begin{cases} x_1^2 \equiv -257, & x_2^2 \equiv -1, & x_3^2 \equiv 16 - x_1x_2, \\ x_4^2 \equiv (-9 + x_1)x_3/(1 - x_2), & (mod \ q). \\ x_5^2 \equiv (-5 + 2x_1)\left(1 + \frac{x_3 - x_2/x_3}{1 - x_2}\right)x_4, \end{cases}$$

(1.2)

Now the system (1.2) is solvable for just those primes $q (> 13)$ which satisfy $q = x^2 + 257y^2$.

In terms of definitions given below, it will be clear that we are constructing cyclic-sixteen class fields of $k_2 = \mathbb{Q}(-p)^{1/2}$ for those primes $p$ for which $h$, the class number of $k_2$, is divisible by 16. In principle, this construction is finitary but not routine (see [1a]); and the generator $\Lambda_0$ is far from unique (in fact, another value is more convenient later in Section 3 below). Yet this construction is especially amenable to com-
puters because, as we shall see, once a correct guess is made, it is sufficient to test three mechanically chosen primes \( q \) to establish the congruence properties like those just described for \( x^2 + 257y^2 \).

2. The Class Fields. We start with \( \text{Cl} \), the ideal class group of order \( h \) for the field

\[
k_2 = \mathbb{Q}(-p)^{1/2} \quad \text{(prime) } p \equiv 1 \pmod{8}.
\]

The 2-Sylow subgroup \( \text{Cl}_2 \) is known to be cyclic \( C(2^T) \), for some \( T \geq 2 \). We call the \( 2^m \)-class group \((0 < m \leq T)\) the subgroup \( \text{Cl}^{2^m} \) of \( \text{Cl} \) consisting of those classes of \( \text{Cl} \) which are \( 2^m \)-powers; then the even part of the \( 2^m \)-class group is \( C(2^{T-m}) \).

The \( 2^m \)-class field \( k_{2m+1} \) is defined uniquely as that normal extension of \( k_2 \) for which a prime ideal \( q \) in \( k_2 \) (of prime norm \( q \)) splits completely in \( k_{2m+1} \) precisely when \( q \) belongs to a class in \( \text{Cl}^{2^m} \). Then \( \text{Gal} k_{2m+1}/k_2 = C(\text{Cl}/\text{Cl}^{2^m}) \) and \( [k_{2m+1}: k_2] = 2^m \). Another characterization of \( k_{2m+1} \) is that it is the unique unramified normal extension of \( k_2 \) of degree \( 2^m \).

For notation we use Latin letters for rational integers and Greek for algebraic, while subscripts or German letters denote ideals (always) in \( k_2 \), e.g., \( (2) = 2^2 \), \( (\epsilon) = \epsilon_1 \epsilon_2 \), etc. We summarize an earlier paper which goes as far as \( k_{16} \), (see [2]). For \( \text{Cl}^2 \) we have genus theory, and

\[
k_4 = k_2(i).
\]

For \( \text{Cl}^4 \) we have

\[
k_8 = k_4(\epsilon^{1/2}),
\]

where \( \epsilon \) is a fundamental unit of \( \mathbb{Q}(p^{1/2}) \), (see table in [5]),

\[
\begin{align*}
(2.4a) & \quad \epsilon = s + tp^{1/2}, \quad \epsilon' = s - tp^{1/2}, \\
(2.4b) & \quad s^2 - t^2p = -1, \quad s > 0, t > 0.
\end{align*}
\]

Figure 1

Tower of class fields over \( k_2 \)
For Cl^8 (when 8 | h) we have

\[ k_{16} = k_8(\Gamma^{1/2}), \]

where \( \Gamma \) is defined by the relations

\[ -p = f^2 - 2e^2, \quad f \equiv -1 \pmod{4}, \quad e > 0, \]

\[ \Gamma = (f + (-p)^{1/2})e^{1/2}/(1 - i). \]

3. Input Data for Cyclic-Sixteen Class Fields. We continue to define new parameters for when 8 | h. First of all we solve

\[ ew^2 = u^2 + pv^2, \quad v > 0, \quad w > 0, \quad u \equiv fv \pmod{e}. \]

The solvability of this equation follows from the fact that in \( k_2 \) \( (2) = 2^2 \) so \( 2_1 \) is an ideal whose class is of order 2, while by (2.6) \( e = N\epsilon_1 \), where \( \epsilon_1 \) is in a class of order 4. Similarly, \( w = N\omega_1 \) so \( \omega_1 \) is in a class of order 8. The congruence conditions of \( u \) and \( v \) guarantee that \( \epsilon_1^2 | f + (-p)^{1/2} \), while \( \epsilon_1 | u + v(-p)^{1/2} \) (this is important when \( e \) is composite). The actual computation is done by machine after preliminary calculations show that \( v \) cannot always be assumed to be one. For the current run we can take \( v \leq 5 \).

We also need to assign signs to radicals. We begin by arbitrarily assigning signs to

\[ (-p)^{1/2}, i, e^{1/2}, \Gamma^{1/2}, \]

subject to \( p^{1/2} = -(-p)^{1/2}i \) in the computation of \( e \) (see (2.4)) and

\[ e'^{1/2} = i/e^{1/2}. \]

Other radicals are now determined. For example, by squaring both sides,

\[ (1 + si)^{1/2} = (e^{1/2} - e'^{1/2})/(1 - i). \]

Furthermore, if we decompose

\[ p = a^2 + b^2, \quad (\text{odd}) \ a > 0, \ (\text{even}) \ b, \]

we can choose the sign of \( b \) so that for suitable integers, \( z_1 \) and \( z_2 \)

\[ (1 + si) = (a + bi)(z_1 + z_2i)^2, \quad z_1 > 0, \quad z_2 > 0 \]

(note \( z_1^2 + z_2^2 = t \)). This is done by using a double-precision complex square-root of the two fractions \((1 + si)/(a \pm bli)\) to find which one is closer to a Gaussian integer. Therefore,

\[ (a + bi)^{1/2} = (e^{1/2} - e'^{1/2})/(1 - i)(z_1 + z_2i). \]

We finally read in from a table of units [6] the fundamental unit for the Gauss-Pell equation

\[ \Omega_0 = \frac{t_1 + it_2 + (u_1 + iu_2)(a + bi)^{1/2}}{2}, \]
<table>
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<tr>
<th></th>
<th>$\Phi(-p)^{1/2}$</th>
<th>$Q(i)$</th>
<th>$\Phi(p^{1/2})$</th>
<th>$Q(\alpha+bi)^{1/2}$</th>
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</thead>
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<td>z₂</td>
<td>t₁</td>
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<td>1</td>
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<td>1</td>
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<td>10725,88716,</td>
</tr>
</tbody>
</table>

**Table I. Input**
where \((a + bi)^{1/2}\) has a sign already specified by (3.5c). According to general methods of Dirichlet [3] (in analogy with the "ordinary" case (2.4)),

\[(3.7) \quad N_{Q(i)\Omega_0} = ((t_1 + it_2)^2 - (u_1 + iu_2)^2(a + bi))/4 = \pm i.\]

(Often there is a more convenient \(\Omega_1\) in \(Q(a + bi)^{1/2}\) of norm \(i\xi^2, \xi \in Q(i)\), which differs from \(\Omega_0\) by a square factor. Thus when \(p = 257\), we can use \(\Omega_1 = (1 + (1 + 16i)^{1/2})\) instead; see (1.1).)

The entries of Table I are now completely accounted for.

**Conjecture 3.8.** When \(16 \mid h\), the radicand of the 16-class field

\[(3.9) \quad k_{32} = k_{16}(\Lambda^{1/2})\]

may be taken as

\[(3.10) \quad \Lambda = (u + v(-p)^{1/2})\Omega \Gamma^{1/2},\]

where \(\Omega\) is either \(\Omega_0\) or \(i\Omega_0\) (as remains to be determined).

We verify this conjecture for the fourteen \(p < 2000\) where \(16 \mid h\). There either \(h = 16\) and \(C_{16}\) consists only of principal classes, or \(h = 32\) and \(C_{16}\) also contains those equivalent to \(2_1\). Thus, in any case, for \(q = Nq\) and \(q \in C_{16}\), we can write

\[(3.11) \quad f_0q = x^2 + py^2, \quad 16f_0 | h.\]

We must show that for exactly such (large) \(q\) the defining equation for \(\Lambda^{1/2}\) splits modulo \(q\) into 32 factors once we have chosen the right \(\Omega\) (= \(\Omega_0\) or \(i\Omega_0\)).

4. **Galois Group Considerations.** We must have \(k_{32}/k_2\) cyclic and \(k_{32}/Q\) dihedral. Thus, we want (compare [2])

\[(4.1) \quad \text{Gal } k_{32}/Q = \langle \sigma, \tau \rangle a^{16} = \tau^2 = (\sigma \tau)^2 = 1,\]

where \(\sigma\) and \(\tau\) may be chosen as follows:

\[
\begin{align*}
\sigma: & \quad \begin{cases} 
(-p)^{1/2} &\rightarrow (-p)^{1/2}, 
\quad p^{1/2} &\rightarrow -p^{1/2}, 
\quad i &\rightarrow -i, 
\end{cases} \\
& \quad e^{1/2} \rightarrow e^{1'/2}, 
\end{align*}
\]

\[
\Omega \rightarrow \sigma \Omega, \quad \Lambda \rightarrow \Lambda \sigma \Omega / \Omega e^{1/2},
\]

\[
\begin{align*}
\tau: & \quad \begin{cases} 
p^{1/2} &\rightarrow p^{1/2}, 
\quad (-p)^{1/2} &\rightarrow -(p)^{1/2}, 
\quad i &\rightarrow -i, 
\end{cases} \\
& \quad e^{1/2} \rightarrow e^{1'/2}, 
\end{align*}
\]

\[
\Omega \rightarrow \tau \Omega, \quad \Lambda \rightarrow e^{2}w^2e^{1'/2}\tau \sigma \Omega / \Lambda. 
\]

For the operations on \(\Omega\), write \(\alpha\) and \(\beta\) as elements of \(Q(i)\), using \(\alpha'\) and \(\beta'\) to denote conjugates over \(Q\).
\[ \Omega = \alpha + \beta(e^{1/2} - e'^{1/2}), \]
\[ \tau \Omega = \sigma \Omega = \alpha' + \beta'(e^{1/2} + e'^{1/2}), \]
\[ \sigma^2 \Omega = \alpha - \beta(e^{1/2} - e'^{1/2}) = \pm i/\Omega, \]
\[ \sigma^{-1} \Omega = \sigma^3 \Omega = \alpha' - \beta'(e^{1/2} + e'^{1/2}) = \pm i/\sigma^3 \Omega. \]

To verify the Galois group (4.1) requires, first of all, normality:

**Conjecture 4.3.** \( (k_{16} =) k_8(\Gamma^{1/2}) \supset k_8(\Sigma^{1/2}) \supset k_8, \) where

\[ \Sigma = \Omega \sigma e^{1/2}. \]

From this result \( k_{16}(\Lambda^{1/2}) \) is normal over \( \mathbb{Q} \). We see this by listing the conjugates of \( \Sigma \) generated by \( \sigma \) and \( \tau \) (all differing by square factors). Since all conjugates of \( k_{32} \) over \( k_2 \) must be generated by \( \sigma \) and since \( \Lambda^{1/2} \notin k_{16} \) (as implied by Conjecture 3.8), then \( \text{Gal} \ k_{32}/k_2 = C(16) \). Similarly, \( k_8(\Sigma^{1/2})/k_2 \) is cyclic independently of Conjecture 4.3. The more tempting conjecture, \( k_{16} = k_8(\Sigma^{1/2}) \supset k_8 \), seems valid but is not needed for now, (compare Section 7 below).

We shall produce a computer output to simultaneously verify Conjectures 3.8 and 4.3.

5. **The Conductor-Discriminant Theorem.** The radicand \( \Lambda \) was set up as a perfect (ideal) square as the first step in finding an unramified \( k_{32} \) over \( k_{16} \) (hence over \( k_2 \)). The worst possible case now is that \( k_{32} \) is ramified over even primes (i.e., \( 2 \)) in \( k_2 \). This would mean, in effect, that for an ideal \( f \) (the conductor) in \( k_2 \), all odd primes in \( k_2 \) congruent to one another mod\( f \) (see (5.1a) below) split completely if one such prime does from \( k_2 \) to \( k_{32} \). This reduces the testing to a finite set; see \([4]\).

**Lemma 5.1.** Let \( K \supset K_1 \supset k \), where \( \text{Gal} \ K/k = C(2^m) \), \( \text{Gal} \ K_1/k = C(2^{m-1}) \); and let \( K_1/k \) be unramified, while \( K = K_1(\Lambda^{1/2}) \), where \( \Lambda \) is an ideal square in \( K_1 \). Then the conductor of \( K/k \) is a divisor of 4. Thus, if \( \wp_1 \) and \( \wp_2 \) are two odd prime ideals in \( k \), they will factor alike in \( K/k \) when they belong to the same class (mod\( \chi \)) in \( k \).

The proof follows from the fact that the different of \( K_1/k \) is 1 (unramified), while that of \( K/K_1 \) divides 2 (since \( \Lambda \) is an ideal square). Thus, the discriminant of \( K/k \) divides \( 2^{2m} \). But by the conductor-discriminant theorem (see Hasse \([4]\)), this discriminant is \( 2 \prod \chi f_\chi \), where \( \chi \) are the characters of \( H_0 = \text{Gal} \ K/k \) and \( f_\chi \) is the conductor over \( k \) of the field fixed by that subgroup of \( H_0 \) for which \( \chi = 1 \). In effect, \( f_\chi = 1 \) for all proper subfields and \( f_\chi \) is the conductor for \( K \) occurring as often in the product as \( \chi \) is primitive, i.e., \( \phi(2^m) = 2^{m-1} \) times. But \( 2^{2m} = 4^{\phi(2^m)} \).

We, therefore, need a refinement of \( \text{Cl}^{2m} \) to \( \text{Cl}^{2m} \) (mod\( \chi \)). Here we consider only odd ideals \( a \) and \( b \); they are equivalent exactly when for odd integers in \( k_2 \), namely \( \alpha \) and \( \beta \)

\[ \alpha a = \beta b, \quad \alpha \equiv \beta \pmod{4}. \]
The even part of $\text{Cl}^{2m}(\mathbb{Z}/4\mathbb{Z})$ is $C(2^{2m}) \times C(2) \times C(2)$. The cycles $C(2) \times C(2)$ come from the four-group of odd principal ideals $(\alpha)$ modulo 4, i.e., $\pm \alpha$, where

$$(5.1b) \quad \alpha \equiv 1, \quad 1 + 2(-p)^{1/2}, \quad (-p)^{1/2}, \quad (-p)^{1/2} + 2 \pmod{4}.$$ 

Once we verify the splitting properties in $\text{Cl}^{16}(\mathbb{Z}/4\mathbb{Z})$ in $k_{32}/k_2$ it will follow (from the equivalent definitions of class field in Section 2) that $k_{32}/k_2$ is unramified and the conductor $f$ was actually the unit ideal.

**Preliminary Computational Procedure 5.2.** For any $p$ (with $16 | h$) we can verify Conjecture 4.3 by testing to see that primes generating $\text{Cl}^8(\mathbb{Z}/4\mathbb{Z})$ split completely in $k_8(\mathbb{Z}^{1/2})$. To verify Conjecture 3.8 we need only have to assume Conjecture 4.3 and make tests to show that primes generating $\text{Cl}^8(\mathbb{Z}/4\mathbb{Z})$ split completely in $k_{16}(\Lambda^{1/2})$ while one prime which splits in $k_{16}$ (i.e., an eighth-power class) does not, (so $\Lambda^{1/2} \notin k_{16}$).

We begin with $\text{Cl}^8$. For given $p$, let $x$ and $y$ vary so as to generate primes $q$ such that

$$(5.3) \quad f_0q = x^2 + py^2, \quad x > 0, y > 0,$$

where $f_0 = 1$ and 2 when $h = 16$ and $f_0 = 1, 2$, and $e$ when $h = 32$. When $f_0 = e$, we further require

$$(5.4) \quad f_0y \equiv \pm x \pmod{e},$$

so for some choice of sign $q \sim e_1^{-1}$ (compare (3.1)). In all cases the class of $q$ is an eighth power, and together they generate $\text{Cl}^8$.

**Final Computational Procedure 5.5.** Select three primes $q$ for each $p$ as follows: Two of them are principal ($f_0 = 1$) and correspond to two of the three non-trivial classes in (5.1b). The third corresponds to a nonprincipal class, namely a generator of $\text{Cl}^8(\mathbb{Z}/4\mathbb{Z})$, (so $f_0 = 2$ when $h = 16$ and $f_0 = e$ when $h = 32$). Procedure 5.2 can be restricted to just these $q$.

The slight improvement from Procedures 5.2 to 5.5 is due to the fact that we really use a multiplicative symbol $\text{tr}((K/k)IC)$ to test the splitting character of the ideal $q$ in class $C$ from $k$ to $K$. Thus, it is trivial that the square of a class will split.

**6. Verification of Conjectures by Output.** The test primes $q$ are chosen by a machine search according to (5.3) (with the a priori guess that $q < 9999$ would suffice). Actually, the machine accepted for output one representative $q$ per class in $\text{Cl}^8(\mathbb{Z}/4\mathbb{Z})$ when available, so Table II was selected from a much longer list.

The arithmetic modulo $q$ was performed with the help of a table of indices generated internally for each $q$. Thus, the machine tried to solve for $x_1, x_2, x_3, x_4, x_5$ representing $(-p)^{1/2}, i, \epsilon^{1/2}, \Gamma^{1/2}, \Lambda^{1/2}$ (as residues modulo a prime divisor of $q$ in $k_{32}$)

$$(6.1) \quad \begin{cases} x_1^2 \equiv -p, & x_2^2 \equiv -1, & x_3^2 \equiv s - tx_1x_2, \\ x_4^2 \equiv (f + x_1)x_3/(1 - x_2), \\ x_5^2 \equiv (u + wx_1)x_2y_4x_4 \equiv w_5, \end{cases} \pmod{q}.$$
To check Conjecture 4, test \( z \neq 0 \) (see (4.4)) and, of course, we let \( i = 1 \) and \( 0 \neq y \neq i = 0 \).

\[
(b \text{ mod } r) = (\epsilon x^y) \equiv \epsilon x^y \equiv a \quad (6.3)
\]

Here \( z \) is represented by \( x^y \), where

\[
(b \text{ mod } r) \left( \frac{(z^y x + i)(z^y - 1)}{(x^y z^y - x^y z^y + 1)} \right) \equiv (z^x)^y \equiv a \quad (6.2)
\]

### Table II: Output

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<th>Index (base ( r ))</th>
<th>[z^x ]</th>
<th>[z^y ]</th>
<th>[z^{xyz} ]</th>
<th>[z^{xyz^y} ]</th>
<th>[z^{xyz^y^y} ]</th>
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</table>

Here \( b \) is represented by \( x^y \).
where $y'_4$ represents $\sigma \Omega$. Thus by (4.2a),

$$(6.4) \quad y'_4 \equiv f(-x_2, x_2/x_3) \pmod{q}.$$ 

The output is given by the indices of $x_1, x_2, x_3, x_4, w_5, w_6$ with primitive root $r \pmod{q-1}$ as shown in Table II. We now have the sign choices of (3.2) in the $x_1, \ldots, x_4$ and the residuacity of $w_5, w_6$. Thus, Procedure 5.5 requires that $w_6$ has an even index, while $w_5$ has an odd index just when $f_0 > 1$.

We use "large" $q$ to avoid $q \mid 2wtp$, so 0 is never a factor in (6.1). If $h = 16 \cdot \text{odd}$ or $32 \cdot \text{odd}$, no modification is required (since our search at worst misses eligible primes $q$ where $f_0 q^{\text{odd}} = x^2 + py^2$). If, however, $64 \nmid h$, we should have to use a different value of $f_0$ in (5.3) to catch the nonprincipal generator of $\text{Cl}^8$, e.g., if $128 \nmid h$, we could take $f_0 = w$.

7. Concluding Remarks. Further computations seem to indicate that when $p \equiv 1 \pmod{4}$, $k_8(\Gamma^{1/2}) = k_8(\Sigma^{1/2}) = k_{16}$, (even when $8 \nmid h$). In fact, it would seem that $k_8$ has as a 2-fundamental system of units

$$(7.1) \quad i, \Omega, \sigma \Omega, e^{1/2}$$

of torsion-free rank 3, although this system becomes no part of a 2-fundamental set in $k_{16}$ (because $\Sigma^{1/2}$ occurs).

The rank of the unit system is an indication of how the current results lead to a much more chaotic state of affairs. It is an easy guess that the 32-class field $k_{64}$ is generated by $\Lambda^{1/2}$, where

$$(7.2) \quad \Lambda^* = (u^* + v^*(-p)^{1/2})\Omega^* \Lambda^{1/2} \Gamma^{-1/2}.$$ 

Here $u^{*2} + v^{*2}p = \sigma w v^2$, as in (3.1), with a similar sign condition to ensure the ideal-square property of $\Lambda^*$. Likewise, $\Omega^*$ is a unit of $k_{16}$ (not $k_8$); and the torsion-free rank of such units is now 7 (not 3). Thus, the chances of guessing $\Omega^*$ become increasingly remote. Nevertheless, the pattern of inductively finding the $2^m$-class field seems, at least conjecturally, clear from (3.10) and (7.2).

As a parallel problem, the criterion for $16 \mid h$ is as yet unknown and seems to be of a much greater degree of difficulty than that of $8 \mid h$, which is given by the representability of $p = a_0^2 + 32b_0^2$; see [1]. The author is greatly indebted to Jeff Lagarias for helpful discussions and speculations as well as comments on the present paper.

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7. K. S. WILLIAMS, "On the divisibility of the class number of $\mathbb{Q}(-p)^{\frac{3}{2}}$ by 16." (Manuscript.)